# Minor Research Project

**File No. 47-901/14(WRO), Date: 20/02/2015**

**(XII Plan)**

**Entitled**

# ‘Study of Certain Classes of Generalized Fibonacci Sequence’

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**By**


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## ***Certificate***

This is to certify that **Mr. M. P. Arvadia** (Assistant Professor) Mathematics Department, of Shri U P Arts, Smt. M G Panchal Science and Shri V L Shah Commerce College, Pilvai has completed his project **File No. : 47-901/14(WRO), Date: 20/02/2015** under UGC Faculty Development Programmed (XII Plan) during February – 2015 to February -2017.

Date:

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Signature of Principal Investigator

Signature of Principal

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**Mansukh P. Arvadia**

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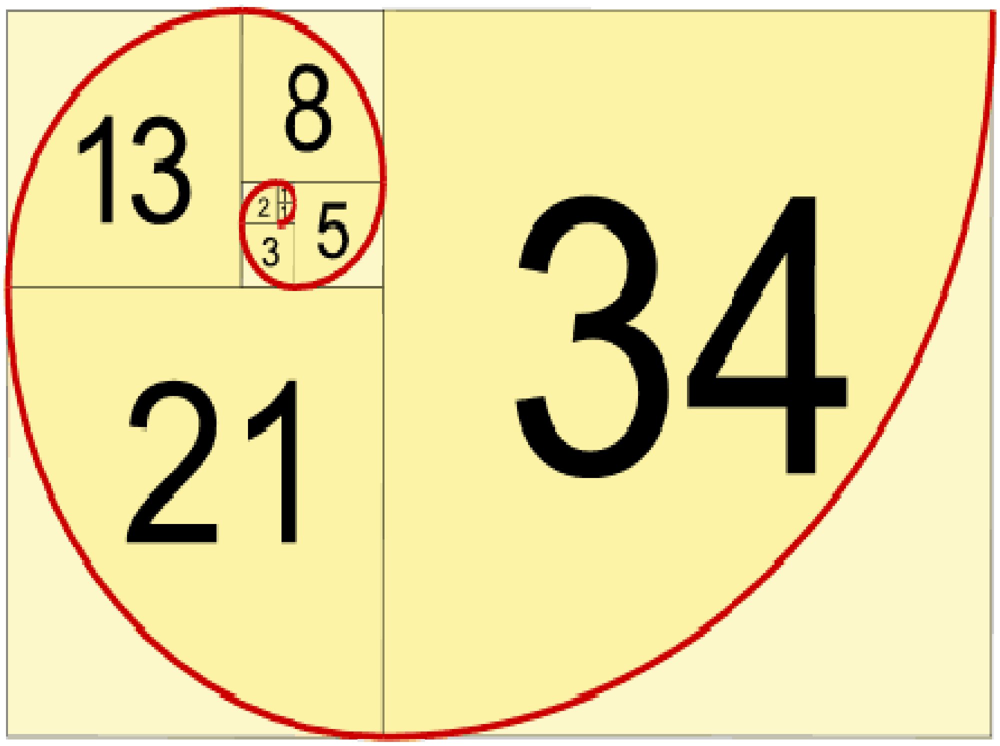
# Chapter: 1




# Introduction

The subject matter of the topic is connected with the study of some aspects of generalization of Fibonacci numbers. The aim of the Research Project is to study various properties of the family of generalized Fibonacci numbers.

In this chapter we present the fundamental symbols, definitions, known facts and some preliminary results from *the Theory of Numbers* and also results connected with elementary properties pursue the essence of the chapters of this research work. The notations and concepts presented here will be used throughout the research work without any further explanation. The proof of some known results mentioned here can be found in Burton [6], Dickson [10], Apostol [2] or any book of elementary Number Theory.

Throughout the thesis, the notations $\sum$ and $\prod$ stands for the usual "summation" and "product" respectively, where the range will be indicated explicitly there. Also by "*induction*" we mean the usual process of mathematical induction.

## 1.1 Preliminaries and Notations:

### 1.1.1 The Divisibility relation:

An integer *a* is said to be divisible by an integer $b \neq 0$, in symbols $b / a$, if there exists some integer *c* such that $a = bc$. In this case we may also say *b divides a*, *b* is a *factor* of *a*, *a* is *divisible* by *b* or *a* is a *multiple* of *b*. If *b* is not a factor of *a* then we write b ∤ a.

**Divisibility properties:** Let *a*, *b*, *c*, *s* and *t* be any integers. Then

(i) If $a / b$ and $b / c$, then $a / c$.

(ii) If $a / b$ and $a / c$, then $a / (sb + tc)$.

(iii) If $a / b$, then $a / bc$.

(iv) If $a / b$ and $c / d$, then $ac / bd$.

### 1.1.2 The Greatest Common Divisor:

If $c / a$ and $c / b$, then we say that *c* is a *common divisor* of *a* and *b*. The *greatest common divisor* (gcd) of two positive integers *a* and *b* is the largest positive integer that divides both *a* and *b* and it is denoted by $\gcd(a, b)$.

Symbolically, we say that a positive integer *d* is the gcd of two positive integers *a* and *b* if (i) $d / a, d / b$ and (ii) if $c / a$, $c / b$ then $c / d$.

## Relatively Prime Integers:

Two positive integers *a* and *b* are said to be relatively prime if $\gcd(a,b)=1$.

An interesting and useful property of gcd is that if $\gcd(a,b)=d$ then $\gcd\left(\frac{a}{d},\frac{b}{d}\right)=1$.

Euclid proved that if $\gcd(a,b)=1$ and if $a/bc$, then $a/c$.

## 1.1.3 The Least Common Multiple:

If $a/c$ and $b/c$ then we say that *c* is a *common multiple* of *a* and *b*. The *least common multiple* (lcm) of two positive integers *a* and *b* is the smallest positive integer which is divisible by both *a* and *b*; it is denoted by $lcm[a,b]$.

Symbolically, a positive integer *m* is the lcm of two positive integers *a* and *b* if (i) $a/m,\ b/m$ and (ii) if $a/n,\ b/n$ then $m/n$.

We note that $gcd(a,b)\times lcm[a,b]=a\times b$ always holds.

## 1.1.4 Congruencies:

Let *m* be a fixed positive integer. An integer *a* is congruent to an integer *b* modulo *m* if $m/(a-b)$. In symbols we write $a\equiv b(\bmod m)$.

Here *m* is said to be modulus of the congruence relation. If *a* is not congruent to *b* modulo *m* then we write $a \not\equiv b(\bmod m)$.

Throughout we assume that all moduli (plural of modulus) are positive integers.

The following properties of congruence always hold:

(1) $a\equiv b(\bmod m)$ if and only if $a=b+mk$ for some integer *k*.

(2) $a\equiv a(\bmod m)$. [Reflexive property]

(3) If $a\equiv b(\bmod m)$ then $b\equiv a(\bmod m)$. [Symmetric property]

(4) If $a\equiv b(\bmod m)$ and $b\equiv c(\bmod m)$ then $a\equiv c(\bmod m)$. [Transitive property]

(5) If $a\equiv b(\bmod m)$ then *a* and *b* leave the same remainder when divided by *m*.

(6) If $a\equiv r(\bmod m)$ where $0\le r<m$ then *r* is the remainder when *a* is divided by *m* and conversely.

By this result it is clear that every integer *a* is congruent to its remainder *r* modulo *m*. Here *r* is said to be *least residue* (or *residue*) of $a(\bmod m)$. Since *r* has exactly *m* choices 0, 1, 2, 3….$(m-1)$ we have the following:

(i) Every integer is congruent (mod m) to exactly one of the least residues

0, 1, 2, 3… $(m-1)$.

(ii) If $a \equiv b(\text{mod}\, m)$ and $c \equiv d(\text{mod}\, m)$, then $a+c \equiv b+d(\text{mod}\, m)$ and

$ac \equiv bd(\text{mod}\, m)$.

With suitable precautions, cancellation can also be allowed, as seen from the following useful result:

(i) If $ca \equiv cb(\text{mod}\, m)$ then $a \equiv b(\text{mod}\, m/d)$, where $d \equiv \gcd(\text{c}, m)$.

Congruence of two integers with different moduli can be combined in to a single congruence as the next result shows.

(ii) If $a \equiv b(\text{mod}\, m_1)$, $a \equiv b(\text{mod}\, m_2)$,…, $a \equiv b(\text{mod}\, m_r)$ then $a \equiv b(mod[m_1, m_2, m_3, ..., m_r])$; where $[m_1, m_2, m_3, ..., m_r]$ is the *lcm* of $m_1, m_2, m_3, ..., m_r$. If they are pair wise relatively prime then this congruence becomes $a \equiv b(mod\; m_1 m_2 m_3 ... m_r)$.

## 1.1.5 Complete System of Residues:

Any set of $m$ integers $a_1, a_2, a_3, ..., a_m$ is said to form *Complete Residue System* (CRS) modulo $m$ if every integer is congruent $(\text{mod}\, m)$ to exactly one $a_r$. In other words, if $a_1, a_2, a_3, ..., a_m$ are congruent modulo $m$ to 0, 1, 2, 3… $(m-1)$ in some order, then we say that $a_1, a_2, a_3, ..., a_m$ constitute CRS$(\text{mod}\, m)$.

## 1.1.6 The Fundamental Theorem of Arithmetic:

Prime numbers are the building blocks of all the integers. Every integer can be decomposed into primes. Before we state this cornerstone result of Theory of Numbers, we need to state the following results:

(i) (Euclid): If $p$ is a prime and $p / ab$, then $p / a$ or $p / b$.

(ii) If $p$ is a prime and $p / a_1 a_2 ... a_n$, then $p / a_i$ for some $i$, where $1 \le i \le n$.

We can now state the most fundamental result in Number Theory.

## Theorem (The Fundamental Theorem of Arithmetic):

Every positive integer $n \ge 2$ is either a prime or can be expressed as a product of primes. The factorization into primes is unique except for the order of the factors.

A factorization of a composite number *n* in terms of primes is a *prime factorization* of *n*. Using the exponential notation, this product can be rewritten in a compact way as $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$; where $p_1, p_2, ..., p_r$ are distinct primes with $p_1 < p_2 < ... < p_r$ and each $\alpha_i$ is a positive integer. This is said to be *prime power representation* of *n*. We can represent this product as $n = \prod p_i^{\alpha_i}$.

We also note that if $p / ab$ and $\gcd(p,a) = 1$ then $p / b$.

### 1.1.7 The Greatest Integer Function:

For an arbitrary real number *x*, we denote by $[x]$ the largest integer less than or equal to *x*; i.e. $[x]$ is the unique integer satisfying $x - 1 < [x] \le x$.

Sometimes $[x]$ is also denoted by $\lfloor x \rfloor$. This function is also known as Floor function.

### 1.1.8 A Generating Function:

The ordinary *generating function* $f(x)$ for the infinite sequence $\{a_0, a_1, a_2, ...\}$ is a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ . Normally the term *generating function* is used to mean ordinary generating function.

Here we usually regard *x* as a place holder rather than a number. Only in a rare cases *x* is regarded as a real number. Thus while actually evaluating a generating function, we can largely forget about questions of convergence.

### 1.1.9 Fermat's Theorem:

The following theorem also known as *Fermat's "Little" theorem* is an important result in the Theory of Numbers.

**Theorem:** If *p* is a prime and $\gcd(p,a) = 1$ then $a^{p-1} \equiv 1 (\bmod p)$.

## 1.2 Introduction to Fibonacci Numbers:

Fibonacci was born in Pisa-Italy around 1170. Around 1192 his father, Guillielmo Bonacci, became director of the Pisan trading colony in Bugia-Algeria, and some time thereafter they traveled together to Bugia. From there Fibonacci traveled throughout Egypt, Syria, Greece, Sicily and Provence where he became familiar with Hindu-Arabic numerals which at that time had not been introduced into Europe.

He returned to Pisa around 1200 and produced *Liber Abaci* in 1202. In it he presented some of the arithmetic and algebra he encountered in his travels, and he introduced the place-valued decimal system and Arabic numerals. Fibonacci continued to write mathematical works at least through 1228, and he gained a reputation as a great mathematician. Not much is known of his life after 1228, but it is commonly held that he died sometime after 1240, presumably in Italy.

Despite his many contributions to mathematics, Fibonacci is today remembered for the sequence which comes from a problem he poses in Liber Abaci. The following is a paraphrase:

A man puts one pair of rabbits in a certain place entirely surrounded by a wall. The nature of these rabbits is such that every month each pair bears a new pair which from the end of their second month on becomes productive. How many pairs of rabbits will there be at the end of one year?

If we assume that the first pair is not productive until the end of the second month, then clearly for the first two months there will be only one pair. At the start of the third month, the first pair will produce a pair giving us a total of two pair. During the fourth month the original pair will produce a pair again but the second pair does not, giving us three pair and so on.

Assuming none of the rabbits die, we can develop a recurrence relation. Let there be $F_n$ pairs of rabbits in month $n$, and $F_{n+1}$ pairs of rabbits in month $n+1$. During month $n+2$, all the pairs of rabbits from month $n+1$ will still be there, and of those rabbits the ones which existed during the $n^{th}$ month will give birth.

Hence $F_{n+2} = F_{n+1} + F_n$.

The sequence which results when $F_1 = F_2 = 1$ is called the Fibonacci sequence and the numbers in the sequence are the Fibonacci numbers:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $9\cdots$ |
| $F_n$ | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | $34\cdots$ |

Thus the answer to Fibonacci problem is 144.

Interestingly, it was not until 1634 that this recurrence relation was written down by Albert Girard.

Despite its simple appearance the Fibonacci sequence $\{F_n\}$ contains a wealth of subtle and fascinating properties which are listed below:

**1.** $\gcd(F_n, F_{n+1}) = 1, \ \forall \ n = 0,1,2,3,\ldots.$.

**2.** $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$.

**3.** $F_{m+n} = F_{m-1}F_n + F_m F_{n+1}$.

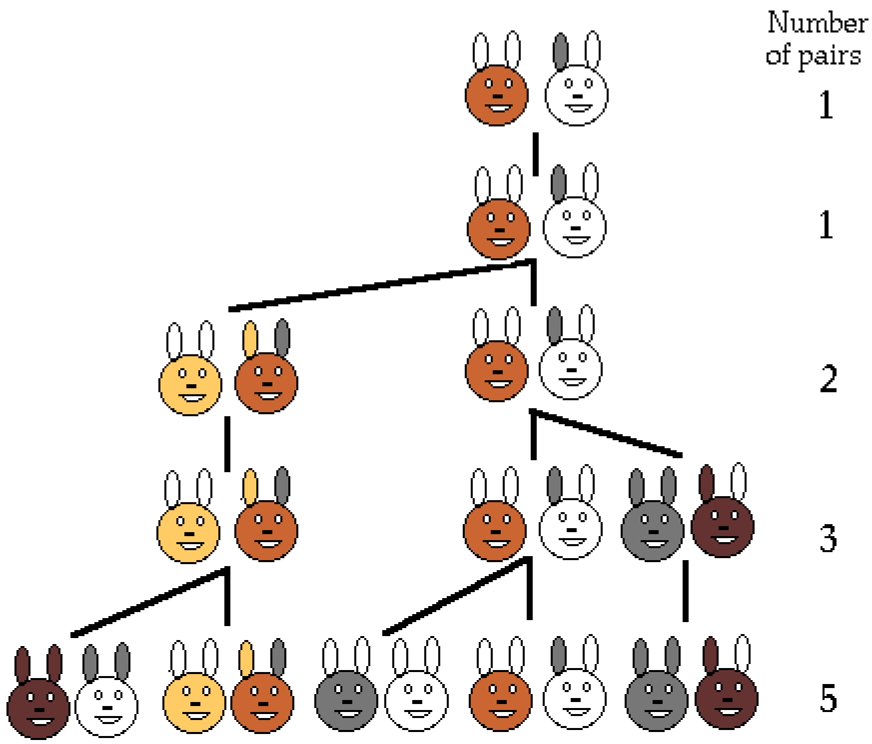
**4.** $F_{-n} = (-1)^{n+1} F_n, \ n \geq 1$.

**5.** $F_{m-n} = (-1)^n (F_m F_{n+1} - F_{m+1} F_n)$

**6.** $F_m \mid F_{mn}$; for all integers $m$ and $n$.

**7.** $F_{n+1}F_{n-1} - F_n^{\,2} = (-1)^n$.

# Chapter- 2




# *Left k-Fibonacci sequence and related identities*

## 2.1 Introduction:

Number Theory is one of the branches of Mathematics related to numbers. The elementary properties are discuss in **[6, 35]** which we used through my research work. Fibonacci sequence $\{F_n\}$ is defined as $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ , for $n \geq 2$ , which gives the sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 …. The Fibonacci numbers also occur in Pascal‘s triangle **[7, 24]**. This sequence arise naturally in many unexpected places and used in equally surprising places like computer algorithms **[5, 18, 19]**, some areas of algebra **[2, 10]**, quasi crystals **[42]** and many areas of mathematics. They occur in a variety of other fields such as finance, art, architecture, music, etc. **[2, 13]** for extensive resources on Fibonacci numbers) The Fibonacci sequence is a source of many identities as appears in the work of Vajda **[43]**. Recently, new generalization of Fibonacci sequences has seized the attention of the mathematicians in **[20, 29, 30, 31]**. The definition of Fibonacci numbers can be extended to define any term as the sum of the preceding three terms i.e. Tribonacci numbers in **[36, 37]**.

There are fundamentally two ways in which the Fibonacci sequence may be generalized; namely, either by maintaining the recurrence relation but altering the first two terms of the sequence from 0, 1 to arbitrary integers *a*, *b* **[29, 38]** or by preserving the first two terms of the sequence but altering the recurrence relation. The two techniques can be combined, but a change in the recurrence relation seems to lead to greater complexity in the properties of the resulting sequence.

We define a generalization of the Fibonacci sequence and call it the generalized Fibonacci sequence. The terms of this sequence are defined by the recurrence relation

$$F_n(a,b) = aF_{n-1}(a,b) + bF_{n-2}(a,b);\ n \geq 2, \qquad (2.1.1)$$

with initial condition $F_0(a,b) = 0$ and $F_1(a,b) = 1$, where $a$ and $b$ are any fixed integers.

The first few terms of this sequence are shown in the following table:

| $n$ | $F_n(a,b)$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | $a$ |
| 3 | $a^2 + b$ |

| | |
|---|---|
| 4 | $a^3+2ab$ |
| 5 | $a^4+3a^2b+b^2$ |

We note that clearly $\{F_n(1,1)\}=\{F_n\}$, the sequence of usual Fibonacci numbers.

## 2.2 The sequence $\{F_n(k,1)\}$, an introduction:

One of the purpose of this chapter is to study the subsequence $\{F_n(k,1)\}$ of $\{F_n(a,b)\}$ by considering $a=k$ and $b=1$ in (2.1.1). We call the sequence $\{F_n(k,1)\}$ as the *left k- Fibonacci sequence* which uses one real parameter *k*. We write it as $F_n(k,1)=F_{k,n}^L$.

**Definition:** For any real number *k*, sequence $\{F_{k,n}^L\}$, the sequence of *left k- Fibonacci numbers* is generated by the recurrence relation

$$F_{k,n}^L=kF_{k,n-1}^L+F_{k,n-2}^L \ ,n\geq 2 \qquad (2.2.1)$$

where $F_{k,0}^L=0$ and $F_{k,1}^L=1$.

In **[8, 11, 27, 44]**, a new generalization of family of Fibonacci sequences and each new choice of *a* and *b* produces a distinct sequence. In **[3, 4, 14, 15, 16, 17]**, the *k*-Fibonacci numbers introduced and give simple proof of an interesting Fibonacci generalization.

Some of the terms of this sequence are shown in the following table:

| $n$ | $F_{k,n}^L$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | $k$ |
| 3 | $k^2+1$ |
| 4 | $k^3+2k$ |
| 5 | $k^4+3k^2+1$ |
| 6 | $k^5+4k^3+3k$ |
| 7 | $k^6+5k^4+6k^2+1$ |
| 8 | $k^7+6k^5+10k^3+4k$ |

| | |
|---|---|
| 9 | $k^8+7k^6+15k^4+10k^2+1$ |
| 10 | $k^9+8k^7+21k^5+20k^3+5k$ |
| 11 | $k^{10}+9k^8+28k^6+35k^4+15k^2+1$ |
| 12 | $k^{11}+10k^9+36k^7+56k^5+35k^3+6k$ |
| 13 | $k^{12}+11k^{10}+45k^8+84k^6+70k^4+21k^2+1$ |
| 14 | $k^{13}+12k^{11}+55k^9+120k^7+126k^5+56k^3+7k$ |
| 15 | $k^{14}+13k^{12}+66k^{10}+165k^8+210k^6+126k^4+28k^2+1$ |
| 16 | $k^{15}+14k^{13}+78k^{11}+220k^9+330k^7+252k^5+84k^3+8k$ |

Appendix- I is computer program to obtain terms of *left k- Fibonacci sequence* $\{F_n\}$ using the programming language MATLAB (R2008a).

- If $k=1$, we get classic Fibonacci sequence defined by $F_0=0, F_1=1$ and $F_n=F_{n-1}+F_{n-2}$, $n\geq 2$. This gives the sequence $\{F_n\}=\{0,1,1,2,3,5,8,13,21,\cdots\}$.
- If $k=2$, we obtain classic Pell's sequence defined by $P_0=0,\ P_1=1$ and $P_n=2P_{n-1}+P_{n-2}$, $n\geq 2$, Here we have $\{P_n\}=\{0,1,2,5,12,29,70,\cdots\}$.
- If $k=3$, we get following sequence defined $H_0=0, H_1=1$ and $H_n=3H_{n-1}+H_{n-2}$, $n\geq 2$. This gives the sequence $\{H_n\}=\{0,1,3,10,33,109,\cdots\}$.

Numerous results are available in the literature for the sequence $\{F_n\}$ of Fibonacci numbers in **[24]**. The simple appearance of the sequence $\{F_{k,n}^L\}$ contains a wealth of subtle and fascinating properties. In this chapter we explore several of the fundamental identities related with sequence $\{F_{k,n}^L\}$.

## 2.3 Some basic identities of left k- Fibonacci numbers:

One of the purposes of this chapter is to develop many of the identities needed in the subsequent chapters. We use the technique of induction as a useful tool in proving many of these identities and theorems involving Fibonacci numbers.

**Lemma 2.3.1** $gcd\,(F_{k,n}^{L}, F_{k,n+1}^{L}) = 1,\ \forall\ n = 0,1,2,3,....$

*Proof*: Suppose that $F_{k,n}^{L}$ and $F_{k,n+1}^{L}$ are both divisible by a positive integer $d$.

Then clearly $F_{k,n+1}^{L} - F_{k,n}^{L} = kF_{k,n}^{L} + F_{k,n-1}^{L} - F_{k,n}^{L} = (k-1)F_{k,n}^{L} + F_{k,n-1}^{L}$ will also be divisible by *d*.

Then right hand side of this result is divisible by *d*. This gives $d \mid F_{k,n-1}^{L}$.

Continuing this argument we see that $d \mid F_{k,n-2}^{L}$, $d \mid F_{k,n-3}^{L}$ and so on. Eventually, we must have $d \mid F_{k,1}^{L}$. Since $F_{k,1}^{L} = 1$, we get $d = 1$.

This proves the required result.

In **[32, 46]** Zeitlin obtain summation formulas and identities for Fibonacci numbers. We denote the sum of first *n* left *k- Fibonacci numbers* by $S_n$ and prove some identities related with the summation of *left k-Fibonacci numbers*.

**Lemma 2.3.2** $S_n = \sum_{i=1}^{n} F_{k,\mathrm{i}}^{L} = \frac{1}{k}(F_{k,n+1}^{L} + F_{k,n}^{L} - 1)$.

*Proof*: We have $F_{k,n}^{L} = kF_{k,n-1}^{L} + F_{k,n-2}^{L},\ \ n \geq 2$. Replacing *n* by 2, 3, 4, … we get

$$F_{k,2}^{L} = kF_{k,1}^{L} + F_{k,0}^{L}$$
$$F_{k,3}^{L} = kF_{k,2}^{L} + F_{k,1}^{L}$$
$$F_{k,4}^{L} = kF_{k,3}^{L} + F_{k,2}^{L}$$
$$\vdots$$
$$F_{k,n-2}^{L} = kF_{k,n-3}^{L} + F_{k,n-4}^{L}$$
$$F_{k,n-1}^{L} = kF_{k,n-2}^{L} + F_{k,n-3}^{L}$$
$$F_{k,n}^{L} = kF_{k,n-1}^{L} + F_{k,n-2}^{L}$$

Now adding all these equations term by term, we get

$$F_{k,2}^{L} + F_{k,3}^{L} + \cdots + F_{k,n}^{L} = F_{k,0}^{L} + (k+1)(\mathrm{F}_{k,1}^{L} + \mathrm{F}_{k,2}^{L} + \cdots + \mathrm{F}_{k,\mathrm{n}-2}^{L}) + \mathrm{k}\mathrm{F}_{k,\mathrm{n}-1}^{L}$$

$$\therefore \quad F_{k,1}^L + F_{k,2}^L + F_{k,3}^L + \cdots + F_{k,n}^L = F_{k,0}^L + F_{k,1}^L + (k+1)(\mathrm{F}_{k,1}^L + \mathrm{F}_{k,2}^L + \cdots + \mathrm{F}_{k,n-2}^L + \mathrm{F}_{k,\mathrm{n}-1}^L + F_{k,n}^L)$$
$$+ \mathrm{k}\mathrm{F}_{k,n-1}^L - (\mathrm{k}+1)(\mathrm{F}_{k,n-1}^L + \mathrm{F}_{k,\mathrm{n}}^L)$$

$$\therefore \quad (1-\mathrm{k}-1)(F_{k,1}^L + F_{k,2}^L + F_{k,3}^L + \cdots + F_{k,n}^L) = F_{k,0}^L + F_{k,1}^L - F_{k,n-1}^L - kF_{k,n}^L - F_{k,n}^L$$

$$\therefore \quad k(F_{k,1}^L + F_{k,2}^L + F_{k,3}^L + \cdots + F_{k,n}^L) = kF_{k,\mathrm{n}}^L + F_{k,\mathrm{n}-1}^L + F_{k,n}^L - F_{k,1}^L$$

$$\therefore \quad F_{k,1}^L + F_{k,2}^L + F_{k,3}^L + \cdots + F_{k,n}^L = \frac{1}{k}(F_{k,n+1}^L + F_{k,n}^L - 1).$$

An alternate method of proving lemma 2.3.2 is to apply the principle of mathematical induction. Using the same process or by induction we can derive formulae for the sum of the first *n left k-Fibonacci numbers* with various subscripts.

We next find the sum of first *n left k-Fibonacci numbers* with only odd or even subscripts.

**Lemma 2.3.3** $\sum_{i=1}^{n} F_{k,2i-1}^L = \frac{1}{k} F_{k,2n}^L$.

*Proof*: We have $F_{k,n}^L = kF_{k,n-1}^L + F_{k,n-2}^L,\ n \ge 2$. Replacing $n$ by 2, 4, 6...., we get

$$F_{k,2}^L = kF_{k,1}^L + F_{k,0}^L$$
$$F_{k,4}^L = kF_{k,3}^L + F_{k,2}^L$$
$$F_{k,6}^L = kF_{k,5}^L + F_{k,4}^L$$
$$\vdots$$
$$F_{k,2n-2}^L = kF_{k,2n-3}^L + F_{k,2n-4}^L$$
$$F_{k,2n}^L = kF_{k,2n-1}^L + F_{k,2n-2}^L.$$

Adding all these equations term by term we get

$$F_{k,2}^L + F_{k,4}^L + F_{k,6}^L + \cdots + F_{k,2n}^L$$
$$= k(F_{k,1}^L + F_{k,3}^L + \cdots + F_{k,2n-1}^L) + (F_{k,0}^L + F_{k,2}^L + F_{k,4}^L + \cdots + F_{k,2n-2}^L)$$
$$= k(F_{k,1}^L + F_{k,3}^L + \cdots + F_{k,2n-1}^L) + (F_{k,2}^L + F_{k,4}^L + \cdots + F_{k,2n-2}^L + F_{k,2n}^L) - F_{k,2n}^L$$

$$\therefore \quad 0 = k(F_{k,1}^L + F_{k,3}^L + \cdots + F_{k,2n-1}^L) - F_{k,2n}^L$$

$\therefore \quad F_{k,1}^L + F_{k,3}^L + F_{k,5}^L + \cdots + F_{k,2n-1}^L = \frac{1}{k} F_{k,2n}^L$. This proves the lemma.

**Lemma 2.3.4** $\sum_{i=1}^{n} F_{k,2n}^{L} = \frac{1}{k}(F_{k,2n+1}^{L} - 1)$.

*Proof*: We have $F_{k,\mathrm{n}}^{L} = kF_{k,\mathrm{n}-1}^{L} + F_{k,\mathrm{n}-2}^{L}, \quad n \geq 2$. Replacing $n$ by 1, 3, 5… we get

$$F_{k,1}^{L} = 1$$

$$F_{k,3}^{L} = kF_{k,2}^{L} + F_{k,1}^{L}$$

$$F_{k,5}^{L} = kF_{k,4}^{L} + F_{k,3}^{L}$$

$$\vdots$$

$$F_{k,2n-1}^{L} = kF_{k,2n-2}^{L} + F_{k,2n-3}^{L}$$

Adding all these equations term by term, we get

$$F_{k,1}^{L} + F_{k,3}^{L} + F_{k,5}^{L} + \cdots + F_{k,2n-1}^{L}$$

$$= 1 + k(F_{k,2}^{L} + F_{k,4}^{L} + \cdots + F_{k,2n-2}^{L}) + (F_{k,1}^{L} + F_{k,3}^{L} + F_{k,5}^{L} + \cdots + F_{k,2n-3}^{L})$$

$$= 1 + k(F_{k,2}^{L} + F_{k,4}^{L} + \cdots + F_{k,2n-2}^{L} + \mathrm{F}_{k,2n}^{L}) - \mathrm{kF}_{k,2n}^{L}$$
$$+ (F_{k,1}^{L} + F_{k,3}^{L} + F_{k,5}^{L} + \cdots + F_{k,2n-1}^{L}) - F_{k,2n-1}^{L}$$

$$\therefore \quad 0 = 1 + k(F_{k,2}^{L} + F_{k,4}^{L} + \cdots + \mathrm{F}_{k,2n}^{L}) - (\mathrm{kF}_{k,2n}^{L} + F_{k,2n-1}^{L}).$$

$$\therefore \quad F_{k,2}^{L} + F_{k,4}^{L} + F_{k,6}^{L} + \cdots + F_{k,2n}^{L} = \frac{1}{k}(F_{k,2n+1}^{L} - 1).$$

The following results follow immediately from above lemmas.

**Corollary 2.3.5** $F_{k,2n}^{L} \equiv 0 \left(mod\ k\right)$ and $F_{k,2n+1}^{L} \equiv 1 \left(mod\ k\right)$.

## 2.4 Some more identities for left k-Fibonacci numbers:

We now derive some more interesting identities for $F_{k,n}^{L}$. First we prove the reduction formula for $F_{k,n}^{L}$.

**Lemma 2.4.1** $F_{k,\mathrm{m+n}}^{L} = F_{k,\mathrm{m}-1}^{L} F_{k,n}^{L} + F_{k,\mathrm{m}}^{L} F_{k,n+1}^{L}$.

*Proof*: Let $m$ be the fixed positive integer. We proceed by inducting on $n$.

For $n = 1$, we have $F_{k,\mathrm{m}+1}^{L} = F_{k,\mathrm{m}-1}^{L} F_{k,1}^{L} + F_{k,\mathrm{m}}^{L} F_{k,2}^{L}$.

Since $F_{k,0}^{L} = 0$, $F_{k,1}^{L} = 1$ and $F_{k,2}^{L} = k$, we have $F_{k,m+1}^{L} = kF_{k,m}^{L} + F_{k,m-1}^{L}$, which is true. This proves the result for $n = 1$.

Now let us assume that the result is true for all integers up to some positive integer '$t$'. Then both

$$F_{k,\mathrm{m}+t}^{L} = F_{k,\mathrm{m}-1}^{L}F_{k,\mathrm{t}}^{L} + F_{k,\mathrm{m}}^{L}F_{k,\mathrm{t}+1}^{L}$$

and $F_{k,\mathrm{m}+(\mathrm{t}-1)}^{L} = F_{k,\mathrm{m}-1}^{L}F_{k,\mathrm{t}-1}^{L} + F_{k,\mathrm{m}}^{L}F_{k,\mathrm{t}}^{L}$ holds.

Now, from these two results we have

$$kF_{k,\mathrm{m}+t}^{L} + F_{k,\mathrm{m}+(\mathrm{t}-1)}^{L} = k(F_{k,\mathrm{m}-1}^{L}F_{k,\mathrm{t}}^{L} + F_{k,\mathrm{m}}^{L}F_{k,\mathrm{t}+1}^{L}) + (F_{k,\mathrm{m}-1}^{L}F_{k,\mathrm{t}-1}^{L} + F_{k,\mathrm{m}}^{L}F_{k,\mathrm{t}}^{L})$$. Thus

$$\begin{aligned}
F_{k,\mathrm{m}+t+1}^{L} &= F_{k,m-1}^{L}(kF_{k,t}^{L} + F_{k,t-1}^{L}) + F_{k,\mathrm{m}}^{L}(\mathrm{k}\,F_{k,t+1}^{L} + F_{k,t}^{L}) \\
&= F_{k,\mathrm{m}-1}^{L}F_{k,t+1}^{L} + F_{k,\mathrm{m}}^{L}F_{k,t+2}^{L} \\
&= F_{k,\mathrm{m}-1}^{L}F_{k,t+1}^{L} + F_{k,\mathrm{m}}^{L}F_{k,(t+1)+1}^{L} = F_{k,\mathrm{m}+(t+1)}^{L},
\end{aligned}$$

This is obviously true.

Thus by the mathematical induction, the result is true for all positive integers $n$.

**Example:** If $m = 2,\ n = 3$ then $LHS = F_{k,2+3}^{L} = F_{k,5}^{L} = k^4 + 3k^2 + 1$ and

$$RHS = F_{k,1}^{L}F_{k,3}^{L} + F_{k,2}^{L}F_{k,4}^{L} = (1)(\mathrm{k}^2+1) + \mathrm{k}(\mathrm{k}^3+2\,\mathrm{k}) = \mathrm{k}^4+3\,\mathrm{k}^2+1.$$

It is often useful to extend the sequence of *left k-Fibonacci numbers* backward with negative subscripts. In fact if we try to extend the *left k-Fibonacci sequence* backwards still keeping to the same rule, we get the following:

| $n$ | $F_{k,\mathrm{n}}^{L}$ |
|---|---|
| -1 | $1$ |
| -2 | $-k$ |
| -3 | $k^2+1$ |
| -4 | $-(\mathrm{k}^3+2\,\mathrm{k})$ |
| -5 | $k^4+3k^2+1$ |
| -6 | $-(\mathrm{k}^5+4\,\mathrm{k}^3+3\,\mathrm{k})$ |
| -7 | $k^6+5k^4+6k^2+1$ |

Thus the sequence of *left k-Fibonacci numbers* is a bilateral sequence, since it can be extended infinitely in both directions. From this table and from the table of values of $F_{k,n}^{L}$, the following result follows immediately.

**Lemma 2.4.2** $F_{k,-\mathrm{n}}^{L} = (-1)^{n+1}F_{k,\mathrm{n}}^{L},\ n \ge 1$.

**Note:** This result will prove later on by Binet's Formula.

We now obtain the extended d'Ocagne's Identity for this sequence.

**Lemma 2.4.3** $F^L_{k,\mathrm{m-n}} = (-1)^n (F^L_{k,\mathrm{m}} F^L_{k,\mathrm{n+1}} - F^L_{k,\mathrm{m+1}} F^L_{k,\mathrm{n}})$.

*Proof*: Replacing $n$ by $-n$ in Lemma 2.4.1, we get $F^L_{k,\mathrm{m-n}} = F^L_{k,\mathrm{m-1}} F^L_{k,-\mathrm{n}} + F^L_{k,\mathrm{m}} F^L_{k,-\mathrm{n+1}}$.

Using the definition of left k-Fibonacci sequence and Lemma 2.4.2, we get

$$F^L_{k,\mathrm{m-n}} = F^L_{k,\mathrm{m-1}} (-1)^{n+1} F^L_{k,\mathrm{n}} + F^L_{k,\mathrm{m}} (-1)^n F^L_{k,\mathrm{n-1}}$$

$$= (-1)^n [F^L_{k,\mathrm{m}} (F^L_{k,\mathrm{n+1}} - kF^L_{k,\mathrm{n}}) - F^L_{k,\mathrm{n}} (F^L_{k,\mathrm{m+1}} - kF^L_{k,\mathrm{m}})]$$

$$= (-1)^n (F^L_{k,\mathrm{m}} F^L_{k,\mathrm{n+1}} - F^L_{k,\mathrm{m+1}} F^L_{k,\mathrm{n}}).$$

We next prove the divisibility property for $F^L_{k,\mathrm{n}}$.

**Lemma 2.4.4** $F^L_{k,m} \mid F^L_{k,mn}$; for any non-zero integers $m$ and $n$.

*Proof*: Let $m$ be any fixed positive integer. We proceed by inducting on $n$.

For $n = 1$, we have $F^L_{k,m} \mid F^L_{k,m}$, which is obvious. This proves the result for $n = 1$.

Now suppose the result is true for all integers $n$ up to some integer '$t$'.

i.e. we assume that $F^L_{k,m} \mid F^L_{k,mt}$.

Then $F^L_{k,m(t+1)} = F^L_{k,mt+m} = F^L_{k,mt-1} F^L_{k,m} + F^L_{k,mt} F^L_{k,m+1}$. But by assumption, we have $F^L_{k,m} \mid F^L_{k,mt}$.

Thus $F^L_{k,m}$ divides the entire right side of the above equation.

Hence $F^L_{k,m} \mid F^L_{k,m(\mathrm{t}+1)}$, which proves the result for all positive integers $n$.

**Note:** By Lemma 2.4.2 it is obvious that the above divisibility criterion holds for negative values of $n$ also.

We now find an expression for the sum of squares of first $n$ *left k-Fibonacci numbers*.

**Lemma 2.4.5** $\sum_{i=1}^{n} F^{L\,2}_{k,\mathrm{i}} = \frac{1}{k} F^L_{k,n} F^L_{k,n+1}$.

*Proof*: We observe that $F^{L\ 2}_{k,\mathrm{m}} = F^L_{k,\mathrm{m}} F^L_{k,\mathrm{m}} = F^L_{k,\mathrm{m}} \left[ \frac{1}{k} \left( F^L_{k,m+1} - F^L_{k,m-1} \right) \right]$

$$= \frac{1}{k} \left( F^L_{k,m} F^L_{k,m+1} - F^L_{k,m} F^L_{k,m-1} \right).$$

Replacing $m$ by 1, 2, 3 … we get

$$F_{k,1}^{L\,2} = \frac{1}{k}(\mathrm{F}_{k,1}^{L}\mathrm{F}_{k,2}^{L})$$

$$F_{k,2}^{L\,2} = \frac{1}{k}(\mathrm{F}_{k,2}^{L}\mathrm{F}_{k,3}^{L} - \mathrm{F}_{k,1}^{L}\mathrm{F}_{k,2}^{L})$$

$$F_{k,3}^{L\,2} = \frac{1}{k}(\mathrm{F}_{k,3}^{L}\mathrm{F}_{k,4}^{L} - \mathrm{F}_{k,2}^{L}\mathrm{F}_{k,3}^{L})$$

$$\vdots$$

$$F_{k,\mathrm{n}-1}^{L\ \ 2} = \frac{1}{k}(\mathrm{F}_{k,\mathrm{n}-1}^{L}\mathrm{F}_{k,\mathrm{n}}^{L} - \mathrm{F}_{k,\mathrm{n}-2}^{L}\mathrm{F}_{k,\mathrm{n}-1}^{L})$$

$$F_{k,\mathrm{n}}^{L\,2} = \frac{1}{k}(\mathrm{F}_{k,\mathrm{n}}^{L}\mathrm{F}_{k,\mathrm{n}+1}^{L} - \mathrm{F}_{k,\mathrm{n}-1}^{L}\mathrm{F}_{k,\mathrm{n}}^{L}).$$

Adding all these equations, we get the required result $\sum_{i=1}^{n} F_{k,\mathrm{i}}^{L\,2} = \frac{1}{k} F_{k,n}^{L} F_{k,n+1}^{L}$.

**Example:** If $n = 3$ then $RHS = \frac{1}{k} F_{k,3}^{L} F_{k,4}^{L} = \frac{1}{k}(k^2+1)(k^3+2k) = \mathrm{k}^4 + 3\,\mathrm{k}^2 + 2$ and

$$LHS = \sum_{i=1}^{3} F_{k,\mathrm{i}}^{L\,2} = F_{k,1}^{L\,2} + F_{k,2}^{L\,2} + F_{k,3}^{L\,2} = 1 + k^2 + (k^2+1)^2 = k^4 + 3k^2 + 2.$$

From this Lemma, the following result follows immediately.

**Corollary 2.4.6** The product of two consecutive *left k- Fibonacci numbers* is given by

$$F_{k,n}^{L} F_{k,n+1}^{L} = k \sum_{i=1}^{n} F_{k,i}^{L\,2}.$$

We next find the value of sum of squares of any two consecutive *left k-Fibonacci numbers*.

**Lemma 2.4.7** $F_{k,\mathrm{n}}^{L\,2} + F_{k,n+1}^{L\ \ 2} = \mathrm{F}_{k,2\mathrm{n}+1}^{L}$.

*Proof*: We prove this result by the principal of mathematical induction.

For $n = 1$, we have $F_{k,1}^{L\,2} + F_{k,2}^{L\,2} = \mathrm{F}_{k,3}^{L}$ and $\mathrm{F}_{k,3}^{L} = 1 + k^2$. This proves the result for $n = 1$.

We assume that result is true for all positive integers up to some positive integer '$t$'.

Thus $F_{k,t}^{L\,2} + F_{k,t+1}^{L\ \ 2} = \mathrm{F}_{k,2t+1}^{L}$ holds by assumption.

Now $F_{k,t+1}^{L\ \ 2} + F_{k,t+2}^{L\ \ 2} = \mathrm{F}_{k,t+1}^{L\ \ 2} + (k\,\mathrm{F}_{k,t+1}^{L} + \mathrm{F}_{k,t}^{L})^2$

$$= F_{k,t+1}^{L\ \ 2} + k^2\,\mathrm{F}_{k,t+1}^{L\ \ 2} + 2k\,\mathrm{F}_{k,t}^{L}\mathrm{F}_{k,t+1}^{L} + F_{k,t}^{L\,2}$$

$$= k^2 {F^L_{k,t+1}}^2 + k\,\mathrm{F}^L_{k,t}\mathrm{F}^L_{k,t+1} + k\,\mathrm{F}^L_{k,t}\mathrm{F}^L_{k,t+1} + \mathrm{F}^L_{k,2t+1}$$

$$= kF^L_{k,t+1}(k\,\mathrm{F}^L_{k,t+1} + F^L_{k,t}) + kF^L_{k,t}F^L_{k,t+1} + F^L_{k,2t+1}$$

$$= kF^L_{k,t+1}\,\mathrm{F}^L_{k,t+2} + kF^L_{k,t+1}F^L_{k,t} + \mathrm{F}^L_{k,2t+1}$$

$$= k(F^L_{k,t+1}\,\mathrm{F}^L_{k,t+2} + F^L_{k,t+1}F^L_{k,t}) + \mathrm{F}^L_{k,2t+1}$$

$$= k(F^L_{k,t+1}F^L_{k,t} + F^L_{k,t+1}\,\mathrm{F}^L_{k,t+2}) + \mathrm{F}^L_{k,2t+1}$$

$$= kF^L_{k,2t+2} + F^L_{k,2t+1} = F^L_{k,2t+3} = F^L_{k,2(t+1)+1}.$$

This proves the result by induction.

**Example:** If $n = 3$ then $LHS = {F^L_{k,3}}^2 + {F^L_{k,4}}^2 = (k^2+1)^2 + (k^3+2k)^2$

$$= k^6 + 5k^4 + 6k^2 + 1 = F^L_{k,7} = RHS.$$

We next derive a result which connects three consecutive *left k- Fibonacci numbers* with odd subscript.

**Lemma 2.4.8** $F^L_{k,2n+5} - (k^2+2)F^L_{k,2n+3} + F^L_{k,2n+1} = 0$.

*Proof*: By definition, $F^L_{k,2n+5} = kF^L_{k,2n+4} + F^L_{k,2n+3} = k(kF^L_{k,2n+3} + F^L_{k,2n+2}) + F^L_{k,2n+3}$

$$= (k^2+1)F^L_{k,2n+3} + kF^L_{k,2n+2}.$$

Now $F^L_{k,2n+5} - (k^2+2)F^L_{k,2n+3} + F^L_{k,2n+1}$

$$= (k^2+1)F^L_{k,2n+3} + kF^L_{k,2n+2} - (k^2+2)F^L_{k,2n+3} + F^L_{k,2n+1}$$

$$= (k^2+1)F^L_{k,2n+3} - (k^2+2)F^L_{k,2n+3} + F^L_{k,2n+3} = 0$$

We finally prove the analogous of one of the oldest identities involving the Fibonacci numbers - Cassini's identity, which was discovered in 1680 by a French astronomer Jean – Dominique Cassini. (**Koshy [24]**)

**Lemma 2.4.9** $F^L_{k,\mathrm{n}+1}F^L_{k,\mathrm{n}-1} - {F^L_{k,\mathrm{n}}}^2 = (-1)^n$.

*Proof*: We have $F^L_{k,\mathrm{n}+1}F^L_{k,\mathrm{n}-1} - {F^L_{k,\mathrm{n}}}^2 = (kF^L_{k,\mathrm{n}} + F^L_{k,\mathrm{n}-1})F^L_{k,\mathrm{n}-1} - {F^L_{k,\mathrm{n}}}^2$

$$= kF^L_{k,n}F^L_{k,n-1} - {F^L_{k,n}}^2 + {F^L_{k,n-1}}^2$$

$$= F^L_{k,\mathrm{n}}(kF^L_{k,\mathrm{n}-1} - F^L_{k,\mathrm{n}}) + {F^L_{k,\mathrm{n}-1}}^2$$

$$= -F^L_{k,\mathrm{n}}F^L_{k,\mathrm{n}-2} + {F^L_{k,\mathrm{n}-1}}^2 \quad = (-1)(F^L_{k,\mathrm{n}}F^L_{k,\mathrm{n}-2} - {F^L_{k,\mathrm{n}-1}}^2).$$

Repeating the same process successively for right side, we get

$$\therefore\ F^{L}_{k,\mathrm{n}+1}F^{L}_{k,\mathrm{n}-1} - {F^{L}_{k,\mathrm{n}}}^{2} = (-1)^{1}(F^{L}_{k,\mathrm{n}}F^{L}_{k,\mathrm{n}-2} - {F^{L}_{k,\mathrm{n}-1}}^{2})$$

$$= (-1)^{2}(F^{L}_{k,\mathrm{n}-1}F^{L}_{k,\mathrm{n}-3} - {F^{L}_{k,\mathrm{n}-2}}^{2})$$

$$= (-1)^{3}(F^{L}_{k,\mathrm{n}-2}F^{L}_{k,\mathrm{n}-4} - {F^{L}_{k,\mathrm{n}-3}}^{2})$$

$$\vdots$$

$$= (-1)^{n}(F^{L}_{k,1}F^{L}_{k,-1} - {F^{L}_{k,0}}^{2})$$

$$= (-1)^{n}.$$

In **[25, 28, 34, 44]** matrices were used to discover facts about the Fibonacci sequence. We now demonstrate a close link between matrices and *left k- Fibonacci numbers*. We define an important $2\times 2$ matrix $U = \begin{bmatrix} 0 & 1 \\ 1 & k \end{bmatrix}$, which plays a significant role in discussions concerning *left k- Fibonacci sequence*.

We prove the following results, which will be used later in the chapter.

**Lemma 2.4.10** (1) $U^{2} = kU + I$ (2) $U^{-2} = I - kU^{-1}$.

*Proof*: (1) We have $U^{2} = \begin{bmatrix} 0 & 1 \\ 1 & k \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & k \end{bmatrix} = \begin{bmatrix} 1 & k \\ k & 1+k^{2} \end{bmatrix}$

$$= k\begin{bmatrix} 0 & 1 \\ 1 & k \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= kU + I$$

(2) $U^{2} = \begin{bmatrix} 1 & k \\ k & 1+k^{2} \end{bmatrix} \Rightarrow U^{-2} = \begin{bmatrix} 1 & k \\ k & 1+k^{2} \end{bmatrix}^{-1} = \begin{bmatrix} 1+k^{2} & -k \\ -k & 1 \end{bmatrix}$

and $U^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & k \end{bmatrix}^{-1} = -\begin{bmatrix} k & -1 \\ -1 & 0 \end{bmatrix}$

$$\therefore\ I - kU^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k\begin{bmatrix} k & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1+k^{2} & -k \\ -k & 1 \end{bmatrix} = U^{-2}.$$

This proves the required results.

**Lemma 2.4.11** If $U=\begin{bmatrix}0&1\\1&k\end{bmatrix}$ then $U^n=\begin{bmatrix}F_{k,n-1}^L & F_{k,n}^L\\ F_{k,n}^L & F_{k,n+1}^L\end{bmatrix}$.

*Proof:* We prove this result by using principal mathematical induction.

For $n=1$, we have $U=\begin{bmatrix}F_{k,0}^L & F_{k,1}^L\\ F_{k,1}^L & F_{k,2}^L\end{bmatrix}=\begin{bmatrix}0&1\\1&k\end{bmatrix}$

This proves the result for $n=1$.

Assume that the result is true for some positive integer $n=t$.

Thus $U^t=\begin{bmatrix}F_{k,\text{t}-1}^L & F_{k,\text{t}}^L\\ F_{k,\text{t}}^L & F_{k,\text{t}+1}^L\end{bmatrix}$ holds by assumption.

Now $U^{t+1}=U^tU=\begin{bmatrix}F_{k,\text{t}-1}^L & F_{k,\text{t}}^L\\ F_{k,\text{t}}^L & F_{k,\text{t}+1}^L\end{bmatrix}\begin{bmatrix}0&1\\1&k\end{bmatrix}=\begin{bmatrix}F_{k,\text{t}}^L & kF_{k,\text{t}}^L+F_{k,t-1}^L\\ F_{k,\text{t}+1}^L & kF_{k,t+1}^L+F_{k,\text{t}}^L\end{bmatrix}=\begin{bmatrix}F_{k,\text{t}}^L & F_{k,\text{t}+1}^L\\ F_{k,\text{t}+1}^L & F_{k,\text{t}+2}^L\end{bmatrix}$

Thus the result is true for $n=t+1$ also. This proves the result by induction.

We can apply above Lemma to derive four new identities as the next corollary shows, although they are basically the same.

**Corollary 2.4.12** $F_{k,m+n-1}^L=F_{k,m}^LF_{k,\text{n}}^L+F_{k,m-1}^LF_{k,\text{n}-1}^L$

$$F_{k,m+n}^L=F_{k,m}^LF_{k,\text{n}+1}^L+F_{k,m-1}^LF_{k,\text{n}}^L$$

$$F_{k,m+n}^L=F_{k,m+1}^LF_{k,\text{n}}^L+F_{k,m}^LF_{k,\text{n}-1}^L$$

$$F_{k,m+n+1}^L=F_{k,m+1}^LF_{k,\text{n}+1}^L+F_{k,m}^LF_{k,\text{n}}^L$$

*Proof:* We know that $U^m\times U^n=U^{m+n}$

$$\therefore\begin{bmatrix}F_{k,m-1}^L & F_{k,m}^L\\ F_{k,m}^L & F_{k,m+1}^L\end{bmatrix}\times\begin{bmatrix}F_{k,\text{n}-1}^L & F_{k,\text{n}}^L\\ F_{k,\text{n}}^L & F_{k,\text{n}+1}^L\end{bmatrix}=\begin{bmatrix}F_{k,m+n-1}^L & F_{k,m+n}^L\\ F_{k,m+n}^L & F_{k,m+n+1}^L\end{bmatrix}$$

$$\therefore\begin{bmatrix}F_{k,m-1}^LF_{k,\text{n}-1}^L+F_{k,m}^LF_{k,\text{n}}^L & F_{k,m-1}^LF_{k,\text{n}}^L+F_{k,m}^LF_{k,\text{n}+1}^L\\ F_{k,m}^LF_{k,\text{n}-1}^L+F_{k,m+1}^LF_{k,\text{n}}^L & F_{k,m}^LF_{k,\text{n}}^L+F_{k,m+1}^LF_{k,\text{n}+1}^L\end{bmatrix}=\begin{bmatrix}F_{k,m+n-1}^L & F_{k,m+n}^L\\ F_{k,m+n}^L & F_{k,m+n+1}^L\end{bmatrix}$$

Now comparing the corresponding entries, we have above results.

**Remark:** Lemma 2.4.9 can be proved by using matrix.

We have $U=\begin{bmatrix}0&1\\1&k\end{bmatrix}$. Then $|U|=-1$. Also $U^n=\begin{bmatrix}F_{k,n-1}^L & F_{k,n}^L\\ F_{k,n}^L & F_{k,n+1}^L\end{bmatrix}$.

Then $\left|U^n\right| = F^L_{k,n-1}F^L_{k,n+1} - F^{L\ 2}_{k,n}$ $\therefore\ \left|U^n\right| = \left[F^L_{k,n-1}F^L_{k,n+1} - F^{L\ 2}_{k,n}\right]$

$\therefore\ \left(-1\right)^n = \left[F^L_{k,n-1}F^L_{k,n+1} - F^{L\ 2}_{k,n}\right]$ $\therefore\ F^L_{k,n-1}F^L_{k,n+1} - F^{L\ 2}_{k,n} = \left(-1\right)^n$.

First, we will describe the terms of the *left k-Fibonacci sequence* $F^L_{k,n}$ explicitly by using a generalization of Binet's formula. Therefore, we will start the main content of the second part by deriving a generalization of Binet's formula (via generating functions).

## 2.5 Generating function for $F^L_{k,n}$:

Generating function provide a powerful technique for solving linear homogenous recurrence relation. Normally generating functions are used in combination with linear recurrence relations with constant coefficients. Here we consider the generating functions for the generalized Fibonacci sequence and derive some of the most fascinating identities satisfied by this sequence.

**Lemma 2.5.1** The generating function for the generalized *left k-Fibonacci sequence* $\left\{F^L_{k,n}\right\}_{n=0}^{\infty}$ is given by $f(x) = \dfrac{x}{1-kx-x^2}$.

*Proof*: We begin with the formal power series representation of generating function for $\left\{F^L_{k,n}\right\}$.

Define $f(x) = \sum_{m=0}^{\infty} F^L_{k,m}x^m = \sum_{m=0}^{\infty} g_m x^m = g_0 + g_1 x + g_2 x^2 + \cdots$

$$= 0 + (1)x + (kg_1 + g_0)x^2 + (kg_2 + g_1)x^3 + (kg_3 + g_2)x^4 + \cdots$$

$$= x + (g_0 + g_1 x + g_2 x^2 + \cdots)x^2 + kx(g_1 x + g_2 x^2 + \cdots)$$

$$= x + x^2 f(x) + kxf(x)$$

$$\therefore\ (1-kx-x^2)f(x) = x \Rightarrow f(x) = \frac{x}{1-kx-x^2}.$$

This is generating function of $\left\{F^L_{k,n}\right\}_{n=0}^{\infty}$.

## 2.6 Extended Binet's formula $F^L_{k,n}$:

While the recurrence relation and initial values determine every term in the Fibonacci sequence, there is an explicit formula for $F_{k,n}^{L}$ which helps to compute any Fibonacci number without using the preceding Fibonacci numbers.

In the 19th century, the French mathematician Jacques Binet derived two remarkable analytical formulas for the Fibonacci and Lucas numbers. In our case, Binet' formula allows us to express the *left k- Fibonacci numbers* in the function of the roots $\alpha$ and $\beta$ of the characteristic equation $x^2 - kx - 1 = 0$, associated to the recurrence relation $F_{k,n}^{L} = F_{k,n-1}^{L} + kF_{k,n-2}^{L}, (n \geq 2)$.

In **[8, 17, 26, 40]**, Binet's Formula was used to obtain some new identities for *k*-Fibonacci numbers. Here we will describe the terms of the *left* $k-$*Fibonacci sequence* $\{F_{k,n}^{L}\}_{n=0}^{\infty}$ explicitly by using a generalization of Binet's formula and then will present extensions of well-known Fibonacci identities such as Cassini's, Catalan's, and d'Ocagne.

Since, we have second order difference equation with constant coefficients. Therefore, it has the characteristic equation $x^2 = kx + 1$. The roots of this equation are $\alpha = \dfrac{k + \sqrt{k^2 + 4}}{2}$ and $\beta = \dfrac{k - \sqrt{k^2 + 4}}{2}$. We note that $\alpha + \beta = k, \alpha\beta = -1$ and $\alpha - \beta = \sqrt{k^2 + 4}$.

**Theorem 2.6.1** The n[th] *left k- Fibonacci number* $F_{k,n}^{L}$ is given by $F_{k,n}^{L} = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$;

where $\alpha, \beta$ are roots of characteristic equation $x^2 = kx + 1$.

*Proof:* Using partial fraction decomposition, we rewrite $f(x)$ as

$$f(x) = \frac{x}{(1-\alpha x)(1-\beta x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

Solving this equation for A and B, we get $A = \dfrac{1}{\alpha - \beta}$, $B = \dfrac{-1}{\alpha - \beta}$

$$f(x) = \frac{1}{\alpha-\beta}\left[\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right] = \frac{1}{\alpha-\beta}\left[(1-\alpha x)^{-1} - (1-\beta x)^{-1}\right]$$

$$= \frac{1}{\alpha-\beta}[(1+\alpha x+\alpha^2 x^2+\cdots)-(1+\beta x+\beta^2 x^2+\cdots)]$$

$$= \frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}(\alpha^n-\beta^n)x^n = \sum_{n=0}^{\infty}\frac{\alpha^n-\beta^n}{\alpha-\beta}x^n = \sum_{n=0}^{\infty}g_n x^n$$

Since we have $g_n = \dfrac{\alpha^n-\beta^n}{\alpha-\beta}$, we have $F^L_{k,n} = \dfrac{\alpha^n-\beta^n}{\alpha-\beta}$.

This is an extended Binet's Formula for $\left\{F^L_{k,n}\right\}_{n=0}^{\infty}$.

**Remark:** We can prove the Lemma- 2.4.2 by using Binet's formula. We have $F^L_{k,n} = \dfrac{\alpha^n-\beta^n}{\alpha-\beta}$.

Substituting $-n$ in place of $n$, we get $F^L_{k,-n} = \dfrac{\alpha^{-n}-\beta^{-n}}{\alpha-\beta} = \dfrac{-\left(\alpha^n-\beta^n\right)}{\alpha^n\beta^n\left(\alpha-\beta\right)}$.

Hence $F^L_{k,-\mathrm{n}} = (-1)^{n+1}F^L_{k,\mathrm{n}}$, $n \geq 1$, as required.

Catalan's Identity for Fibonacci numbers was found in 1879 by Eugene Charles Catalan a Belgian mathematician who worked for the Belgian Academy of Science in the field of Number Theory. Here we obtain analogues result.

**Lemma 2.6.2** $F^L_{k,n-r}F^L_{k,n+r} - F^{L\ 2}_{k,n} = (-1)^{n+1-r}F^{L\ 2}_{k,r}$.

*Proof:* By using theorem 2.6.1 and $\alpha\beta = -1$ on LHS of result, we get

$$LHS = \frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta}\frac{\alpha^{n+r}-\beta^{n+r}}{\alpha-\beta} - \left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)^2$$

$$= \frac{\left(\alpha^{n-r}-\beta^{n-r}\right)\left(\alpha^{n+r}-\beta^{n+r}\right)-\alpha^{2n}+2\alpha^n\beta^n-\beta^{2n}}{\left(\alpha-\beta\right)^2}$$

$$= \frac{1}{\left(\alpha-\beta\right)^2}\left[-\left(\alpha\beta\right)^{n-r}\beta^{2r}-\left(\alpha\beta\right)^{n-r}\alpha^{2r}+2\left(\alpha\beta\right)^n\right]$$

$$= \frac{1}{\left(\alpha-\beta\right)^2}\left[-\left(-1\right)^{n-r}\beta^{2r}-\left(-1\right)^{n-r}\alpha^{2r}+2\left(-1\right)^n\right]$$

$$= \frac{\left(-1\right)^{n-r+1}}{\left(\alpha-\beta\right)^2}\left[\beta^{2r}+\alpha^{2r}+2\left(-1\right)^{r-1}\right]$$

$$= \frac{\left(-1\right)^{n-r+1}}{\left(\alpha-\beta\right)^2}\left[\beta^{2r}+\alpha^{2r}-2\left(-1\right)^{r}\right]$$

$$=\frac{(-1)^{n-r+1}}{(\alpha-\beta)^2}\left[\beta^{2r}+\alpha^{2r}-2(\alpha\beta)^r\right]$$

$$=(-1)^{n-r+1}\frac{(\alpha^r-\beta^r)^2}{\alpha-\beta}=(-1)^{n-r+1}F_{k,r}^{L\ 2}.$$

This completes the proof.

The following is analogues to one of the oldest identities involving the Fibonacci numbers, which was discovered in 1680 by Jean-Dominique Cassini, a French Astronomer.

**Lemma 2.6.3** $F_{k,n-1}^{L}F_{k,n+1}^{L}-F_{k,n}^{L\ 2}=(-1)^n$.

*Proof:* Taking $r=1$ in Catalan's Identity gives Cassini's Identity for the *left k-Fibonacci numbers*.

Now we derive extended d'Ocagne's Identity.

**Lemma 2.6.4** If $m>n$, we have $F_{k,\mathrm{m}}^{L}F_{k,n+1}^{L}-F_{k,\mathrm{m}+1}^{L}F_{k,n}^{L}=(-1)^n F_{k,\mathrm{m-n}}^{L}$.

*Proof:* By using theorem 2.6.1 and $\alpha\beta=-1$ on *LHS* of the required result we get

$$LHS=\frac{\alpha^m-\beta^m}{\alpha-\beta}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}-\frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta}\frac{\alpha^n-\beta^n}{\alpha-\beta}$$

$$=\frac{1}{(\alpha-\beta)^2}\left[(\alpha^m-\beta^m)(\alpha^{n+1}-\beta^{n+1})-(\alpha^{m+1}-\beta^{m+1})(\alpha^n-\beta^n)\right]$$

$$=\frac{1}{(\alpha-\beta)^2}\left[\alpha\beta(\alpha^m\beta^{n+1})+\alpha\beta(\alpha^{n+1}\beta^m)+\alpha^{m+1}\beta^n+\alpha^n\beta^{m+1}\right]$$

$$=\frac{(\alpha\beta)^n}{(\alpha-\beta)^2}\left[\alpha^{m-n+1}\beta^2+\alpha^2\beta^{m-n+1}+\alpha^{m-n+1}+\beta^{m-n+1}\right]$$

$$=\frac{(-1)^n}{(\alpha-\beta)^2}(\alpha\beta)^2\left[\frac{\alpha^{m-n}}{\alpha}+\frac{\beta^{m-n}}{\beta}+\frac{\alpha^{m-n}}{\alpha\beta^2}+\frac{\beta^{m-n}}{\alpha^2\beta}\right]$$

$$=\frac{(-1)^n}{(\alpha-\beta)^2}\left[\frac{\alpha^{m-n}}{\alpha}+\frac{\beta^{m-n}}{\beta}-\frac{\alpha^{m-n}}{\beta}-\frac{\beta^{m-n}}{\alpha}\right]$$

$$=\frac{(-1)^n}{(\alpha-\beta)^2}\left[\alpha^{m-n}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)+\beta^{m-n}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)\right]$$

$$= \frac{(-1)^n}{(\alpha-\beta)^2}\left[(\beta-\alpha)\frac{\alpha^{m-n}}{\alpha\beta}+(\alpha-\beta)\frac{\beta^{m-n}}{\alpha\beta}\right]$$

$$= (-1)^n\left[\frac{\alpha^{m-n}-\beta^{m-n}}{\alpha-\beta}\right] = (-1)^n F^L_{k,m-n}$$

**Lemma 2.6.5** The sum of first $(n+1)$ terms of the *left k -Fibonacci sequence* is given by

$$\sum_{i=0}^{n} F^L_{k,i} = S^L_{k,n} = \frac{1}{k}\left(F^L_{k,n+1}+F^L_{k,n}-1\right)$$

*Proof:* Using theorem 2.6.1, we write $S^L_{k,n}$ as $S^L_{k,n} = \frac{1}{\alpha-\beta}\sum_{i=0}^{n}(\alpha^i-\beta^i) = \frac{1}{\alpha-\beta}\sum_{i=0}^{n}\alpha^i - \sum_{i=0}^{n}\beta^i$

$$= \frac{1}{\alpha-\beta}\left(\frac{\alpha^{n+1}-1}{\alpha-1}-\frac{\beta^{n+1}-1}{\beta-1}\right)$$

$$= \frac{\alpha^{n+1}\beta-\beta-\alpha^{n+1}+1-\alpha\beta^{n+1}+\alpha+\beta^{n+1}-1}{(\alpha-\beta)(\alpha-1)(\beta-1)}$$

$$= \frac{-\alpha^n+\beta^n+\alpha-\beta-\alpha^{n+1}+\beta^{n+1}}{(\alpha-\beta)(\alpha-1)(\beta-1)}$$

$$= \frac{-(\alpha^n-\beta^n)+(\alpha-\beta)-(\alpha^{n+1}-\beta^{n+1})}{(\alpha-\beta)(-k)}$$

Hence

$$S^L_{k,n} = \frac{1}{k}\left[\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}+\frac{\alpha^n-\beta^n}{\alpha-\beta}-1\right] = \frac{1}{k}\left[F^L_{k,n+1}+F^L_{k,n}-1\right]$$

We now prove a simple but important result which states that the limit of the quotient of two consecutive terms of $\{F^L_{k,n}\}$ is equal to the positive root of the corresponding characteristic equation.

**Lemma 2.6.6** $\lim\limits_{x\to\infty}\frac{F^L_{k,n}}{F^L_{k,n-1}} = \alpha$.

*Proof*: Using Theorem 2.6.1 for $F^L_{k,\mathrm{n}}$ , and using the fact that

$\lim\limits_{x\to\infty}\left(\frac{\beta}{\alpha}\right)^n = 0, \quad |\beta| < \alpha$, we get

$$\lim_{x\to\infty}\frac{F_{k,n}^{L}}{F_{k,n-1}^{L}}=\lim_{x\to\infty}\frac{\alpha^{n}-\beta^{n}}{\alpha^{n-1}-\beta^{n-1}}=\lim_{x\to\infty}\frac{1-\left(\frac{\beta}{\alpha}\right)^{n}}{\frac{1}{\alpha}-\left(\frac{\beta}{\alpha}\right)^{n}.\frac{1}{\beta}}=\alpha .$$

**Note:** For classical Fibonacci sequence we have $\lim_{x\to\infty}\frac{F_n}{F_{n-1}}=\tau$, a golden proportion.

We now prove a combinatorial identity which expresses $F_{k,\mathrm{n}}^{L}$ as an series with finite number of terms.

**Lemma 2.6.7** $F_{k,n}^{L}=\frac{1}{2^{n-1}}\sum_{i=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2i+1}k^{n-1-2i}\left(k^{2}+4\right)^{i}$.

*Proof* By theorem 2.6.1, we have

$$F_{k,n}^{L}=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}=\frac{1}{\alpha-\beta}\left[\left(\frac{k+\sqrt{k^{2}+4}}{2}\right)^{n}-\left(\frac{k-\sqrt{k^{2}+4}}{2}\right)^{n}\right]$$

$$=\frac{1}{2^{n}\sqrt{k^{2}+4}}\left[\left(k+\sqrt{k^{2}+4}\right)^{n}-\left(k-\sqrt{k^{2}+4}\right)^{n}\right]$$

$$=\frac{1}{2^{n}\sqrt{k^{2}+4}}\left\{\begin{array}{l}\left[k^{n}+\binom{n}{1}k^{n-1}\sqrt{k^{2}+4}+\binom{n}{2}k^{n-2}\left(\sqrt{k^{2}+4}\right)^{2}+\cdots\right]\\-\left[k^{n}-\binom{n}{1}k^{n-1}\sqrt{k^{2}+4}+\binom{n}{2}k^{n-2}\left(\sqrt{k^{2}+4}\right)^{2}-\cdots\right]\end{array}\right\}$$

$$=\frac{1}{2^{n}\sqrt{k^{2}+4}}\left[2\binom{n}{1}k^{n-1}\sqrt{k^{2}+4}+2\binom{n}{3}k^{n-3}\left(\sqrt{k^{2}+4}\right)^{3}+\cdots\right]$$

$$=\frac{1}{2^{n-1}}\left[\binom{n}{1}k^{n-1}+\binom{n}{3}k^{n-3}\left(\sqrt{k^{2}+4}\right)^{2}+\binom{n}{5}k^{n-5}\left(\sqrt{k^{2}+4}\right)^{4}+\cdots\right]$$

$$=\frac{1}{2^{n-1}}\left[\binom{n}{1}k^{n-1}+\binom{n}{3}k^{n-3}\left(k^{2}+4\right)+\binom{n}{5}k^{n-5}\left(k^{2}+4\right)^{2}+\cdots\right]$$

$$\therefore\ F_{k,n}^{L}=\frac{1}{2^{n-1}}\sum_{i=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2i+1}k^{n-1-2i}\left(k^{2}+4\right)^{i}.$$

In **[3, 4, 11, 44]** many identities were derived related to $\alpha$ and $\beta$. We next express $\alpha^{n}$ and $\beta^{n}$ in terms of two consecutive values of $F_{k,\mathrm{n}}^{L}$.

**Lemma 2.6.8** $\alpha^n = \alpha F_{k,n}^L + F_{k,n-1}^L$ and $\beta^n = \beta F_{k,n}^L + F_{k,n-1}^L$.

*Proof*: We prove this result by the principal of mathematical induction.

For $n=1$, we have $RHS = \alpha F_{k,1}^L + F_{k,0}^L = \alpha = LHS$

This proves the result for $n=1$.

We assume that result is true for all positive integers up to some positive integers 't'. Thus $\alpha^t = \alpha F_{k,\mathrm{t}}^L + F_{k,\mathrm{t}-1}^L$ holds by assumption.

Now, $\alpha^{t+1} = \alpha\alpha^t = \alpha(\alpha F_{k,t}^L + F_{k,t-1}^t) = \alpha^2 F_{k,t}^t + \alpha F_{k,t-1}^t$

$$= (\alpha F_{k,2}^t + F_{k,1}^t)\mathrm{F}_{k,t}^t + \alpha F_{k,t-1}^t = (\alpha k+1)\mathrm{F}_{k,t}^t + \alpha F_{k,t-1}^t$$

$$= \alpha(kF_{k,t}^L + F_{k,t-1}^L) + F_{k,t}^L = \alpha F_{k,t+1}^L + F_{k,t}^L \qquad \therefore \alpha^{t+1} = \alpha F_{k,t+1}^L + F_{k,t}^L.$$

This proves the result is true for $n = t+1$ also.

Hence the result is true for all positive integers *n*. Second result can be proved on the same line.

We now define $P_n = \alpha^n$; $Q_n = \beta^n$. We first derive the recurrence relation for both $P_n$ and $Q_n$.

**Lemma 2.6.9** $P_n = kP_{n-1} + P_{n-2}$; $n \geq 2$.

*Proof*: We prove this result by the principle of mathematical induction. For $n=2$, we have

$$P_2 = \alpha^2 = \frac{1}{4}\left(k^2 + 2k\sqrt{k^2 + \rightarrow 4} + k^2 + 4\right) = \frac{1}{2}\left(k^2 + k\sqrt{k^2 + \rightarrow 4} + 2\right)$$

$$= k\left(\frac{k+\sqrt{k^2+4}}{2}\right) + 1 = k\alpha + 1 = kP_1 + P_0$$

$P_2 = kP_1 + P_0$, which is true for $n=2$.

We now assume that the result is true for some positive integer *t*.

Thus $P_t = kP_{t-1} + P_{t-2}$ holds.

Now $P_{t+1} = \alpha^{t+1} = \alpha^{t-1}\alpha^2 = \alpha^{t-1}\left(kP_1 + P_0\right) = k\alpha^t + \alpha^{t-1} = kP_t + P_{t-1}$

This proves that result is true for all positive integer *n*.

We also have the following result:

**Lemma 2.6.10** $Q_n = kQ_{n-1} + Q_{n-2}$; $n \geq 2$.

We now prove a result which gives the value of $F_{k,\mathrm{n}}^L$ in terms of power of α.

# Chapter- 3

# Right k-Fibonacci sequence and related identities

## 3.1 The Sequence $\{F_n(1,k)\}$, an Introduction:

In chapter- 2, we defined a generalization of the Fibonacci sequence and call it the generalized Fibonacci sequence. The terms of this sequence are defined by the recurrence relation

$$F_n(a,b) = aF_{n-1}(a,b) + bF_{n-2}(a,b);\ n \geq 2,$$

with initial condition $F_0(a,b)=0$ and $F_1(a,b)=1$; where $a$ and $b$ are any fixed integers.

We note that this generalization is in fact a family of sequences and each new choice of $a$ and $b$ produces a distinct sequence. One of the purpose of this chapter is to study the subsequence $\{F_n(1,k)\}$ of $\{F_n(a,b)\}$ by considering $a=1$ and $b=k$. The sequence $\{F_n(1,k)\}$ is called the *right k-Fibonacci sequence* which uses one real parameter *k*. We write it as $F_n(1,k) = F^R_{k,n}$.

**Definition:** For any real number *k*, sequence $\{F^R_{k,n}\}$, the sequence of *right k- Fibonacci numbers* is generated by the recurrence relation $F^R_{k,n} = F^R_{k,n-1} + kF^R_{k,n-2},\ n \geq 2$;
where $F^R_{k,0} = 0$ and $F^R_{k,1} = 1$.

First few terms of this sequence are shown in the following table:

| $n$ | $F^R_{k,\text{n}}$ |
|---|---|
| 0 | $0$ |
| 1 | $1$ |
| 2 | $1$ |
| 3 | $1+k$ |
| 4 | $1+2k$ |
| 5 | $1+3k+k^2$ |
| 6 | $1+4k+3k^2$ |
| 7 | $1+5k+6k^2+k^3$ |
| 8 | $1+6k+10k^2+4k^3$ |
| 9 | $1+7k+15k^2+10k^3+k^4$ |

10 $1+8k+21k^2+20k^3+5k^4$

Appendix-II is computer program to obtain terms of *right k- Fibonacci sequence* $\{G_n\}$ using the programming language MATLAB (R2008a).

- If $k=1$, we get classic Fibonacci sequence defined by $F_0=0, F_1=1$ and $F_n=F_{n-1}+F_{n-2}$, $n\geq 2$. This gives the sequence $\{F_n\}=\{0,1,1,2,3,5,8,13,21,....\}$
- If $k=2$, we obtain classic Pell's sequence defined by $P_0=0,\ P_1=1$ and $P_n=P_{n-1}+2P_{n-2}$, $n\geq 2$. Here we have $\{P_n\}=\{0,1,1,3,5,11,21,43,85,171,....\}$
- If $k=3$, we get following sequence defined $H_0=0, H_1=1$ and $H_n=H_{n-1}+3H_{n-2}$, $n\geq 2$. This gives the sequence $\{H_n\}=\{0,1,1,4,7,19,40,97,...\}$

## 3.2 Some basic identities of right *k*- Fibonacci numbers:

One of the purposes of this chapter is to develop many of the identities needed in the subsequent chapters. We use the technique of induction as a useful tool in proving many of these identities and theorems involving Fibonacci numbers.

We denote the sum of first $n$ right k- Fibonacci numbers by $S_n$ and prove some identities related with the summation of *right k-Fibonacci numbers*.

**Lemma 3.2.1** $S_n=\sum_{i=1}^{n}F_{k,\mathrm{i}}^R=\frac{1}{k}(F_{k,n+2}^R-1)$.

*Proof*: We have $F_{k,n}^R=F_{k,n-1}^R+kF_{k,n-2}^R,\ \ n\geq 2$. Replacing *n* by 2, 3, 4,… we get

$$F_{k,2}^R=F_{k,1}^R+kF_{k,0}^R$$

$$F_{k,3}^R=F_{k,2}^R+kF_{k,1}^R$$

$$F_{k,4}^R=F_{k,3}^R+kF_{k,2}^R$$

$$\vdots$$

$$F_{k,n-2}^R=F_{k,n-3}^R+kF_{k,n-4}^R$$

$$F_{k,n-1}^R=F_{k,n-2}^R+kF_{k,n-3}^R$$

$$F_{k,n}^R=F_{k,n-1}^R+kF_{k,n-2}^R$$

Now adding all these equations term by term, we get

$$F_{k,2}^{R}+F_{k,3}^{R}+\cdots+F_{k,n}^{R}=(\mathrm{F}_{k,1}^{R}+\mathrm{F}_{k,2}^{R}+\cdots+\mathrm{F}_{k,\mathrm{n}-1}^{R})+\mathrm{k}(\mathrm{F}_{k,0}^{R}+\mathrm{F}_{k,1}^{R}+\mathrm{F}_{k,2}^{R}+\cdots+\mathrm{F}_{k,\mathrm{n}-2}^{R})$$

$$\therefore \quad F_{k,n}^{R}-F_{k,1}^{R}=k(F_{k,0}^{R}+\mathrm{F}_{k,1}^{R}+\mathrm{F}_{k,2}^{R}+\cdots+\mathrm{F}_{k,n}^{R})-k\,\mathrm{F}_{k,n-1}^{R}-\mathrm{k}\,F_{k,n}^{R}$$

$$\therefore \quad k(F_{k,1}^{R}+F_{k,2}^{R}+\cdots+F_{k,n}^{R})=F_{k,n}^{R}+kF_{k,n-1}^{R}+kF_{k,n}^{R}-F_{k,1}^{R}$$

$$=F_{k,n+1}^{R}+kF_{k,n}^{R}-1=F_{k,n+2}^{R}-1$$

$$\therefore \quad F_{k,1}^{R}+F_{k,2}^{R}+F_{k,3}^{R}+\cdots+F_{k,n}^{R}=\frac{1}{k}(F_{k,n+2}^{R}-1).$$

**Lemma 3.2.2** $S_{2n}=\sum_{i=1}^{2n}F_{k,i}^{R}=\frac{1}{k}(F_{k,2n+2}^{R}-1)$.

*Proof*: We have $F_{k,n}^{R}=F_{k,n-1}^{R}+kF_{k,n-2}^{R},\ \ n\geq 2$. Replacing $n$ by 2, 3, 4,… we get

$$F_{k,2}^{R}=F_{k,1}^{R}+kF_{k,0}^{R}$$

$$F_{k,3}^{R}=F_{k,2}^{R}+kF_{k,1}^{R}$$

$$F_{k,4}^{R}=F_{k,3}^{R}+kF_{k,2}^{R}$$

$$\vdots$$

$$F_{k,2n-1}^{R}=F_{k,2n-2}^{R}+kF_{k,2n-3}^{R}$$

$$F_{k,2n}^{R}=F_{k,2n-1}^{R}+kF_{k,2n-2}^{R}$$

Now adding all these equations term by term, we get

$$F_{k,2}^{R}+F_{k,3}^{R}+\cdots+F_{k,2n}^{R}=(1+\mathrm{k})(\mathrm{F}_{k,1}^{R}+\mathrm{F}_{k,2}^{R}+\cdots+\mathrm{F}_{k,2\mathrm{n}-2}^{R})+\mathrm{F}_{k,2\mathrm{n}-1}^{R}$$

$$\mathrm{F}_{k,1}^{R}+\mathrm{F}_{k,2}^{R}+\cdots+\mathrm{F}_{k,2n}^{R}=F_{k,1}^{R}+(1+k)(\mathrm{F}_{k,1}^{R}+\mathrm{F}_{k,2}^{R}+\cdots+\mathrm{F}_{k,2n}^{R})$$
$$-(1+k)(\mathrm{F}_{k,2n-1}^{R}+F_{k,2n}^{R})+F_{k,2n-1}^{R}$$

$$\therefore \quad k(F_{k,1}^{R}+F_{k,2}^{R}+\cdots+F_{k,2n}^{R})=F_{k,2n}^{R}+kF_{k,2n-1}^{R}+kF_{k,2n}^{R}-F_{k,1}^{R}$$

$$=F_{k,2n+1}^{R}+kF_{k,2n}^{R}-1=F_{k,2n+2}^{R}-1 \;\therefore \quad F_{k,1}^{R}+F_{k,2}^{R}+F_{k,3}^{R}+\cdots+F_{k,2n}^{R}=\frac{1}{k}(F_{k,2n+2}^{R}-1).$$

The following results follow immediately from above two lemmas.

**Corollary 3.2.3** $F_{k,n+2}^{R}\equiv 1\left(mod\ k\right)$ and $F_{k,2n+2}^{R}\equiv 1\left(mod\ k\right)$.

*Proof:* We use Mathematical Induction to prove the first result.

For $n=1$, we have $F_{k,1+2}^{R}=F_{k,3}^{R}=1+k\equiv 1(mod\,k)$.

We now assume that the result is true for some positive integer $n=r$. Thus by assumption, $F_{k,r+2}^{R}\equiv 1(mod\,k)$ holds.

Now $F_{k,r+1+2}^{R}=F_{k,r+3}^{R}=F_{k,r+2}^{R}+kF_{k,r+1}^{R}\equiv 1(mod\,k)$.

So the result is true for $n=r+1$ also. This proves the result for all integers *n*.

The second result can be proved on the same line.

An alternate method of proving Lemma 3.2.1 is to apply the principle of mathematical induction. Using the same process or by induction we can derive formulae for the sum of the first *n right k-Fibonacci numbers* with various subscripts.

We next find the sum of first *n right k-Fibonacci numbers* with only odd or even subscripts.

**Lemma 3.2.4** $\sum_{i=1}^{n}F_{k,2i-1}^{R}=\frac{1}{k(2-k)}(F_{k,2n+2}^{R}-kF_{k,2n+1}^{R}+k-1)$.

*Proof*: We have $F_{k,n}^{R}=F_{k,n-1}^{R}+kF_{k,n-2}^{R},\ \ n\geq 2$. Replacing *n* by 3, 5, 7 … we get

$$F_{k,3}^{R}=F_{k,2}^{R}+kF_{k,1}^{R}$$

$$F_{k,5}^{R}=F_{k,4}^{R}+kF_{k,3}^{R}$$

$$F_{k,7}^{R}=F_{k,6}^{R}+kF_{k,5}^{R}$$

$$\vdots$$

$$F_{k,2n-1}^{R}=F_{k,2n-2}^{R}+kF_{k,2n-3}^{R}$$

Adding all these equations term by term and using Lemma 3.2.2, we get

$$F_{k,1}^{R}+F_{k,3}^{R}+F_{k,5}^{R}+\cdots+F_{k,2n-1}^{R}=F_{k,1}^{R}+(F_{k,2}^{R}+F_{k,4}^{R}+\cdots+F_{k,2n-2}^{R})$$
$$+k(F_{k,1}^{R}+F_{k,3}^{R}+\cdots+F_{k,2n-3}^{R})$$

$$\therefore\ 2(F_{k,1}^{R}+F_{k,3}^{R}+F_{k,5}^{R}+\cdots+F_{k,2n-1}^{R})=1+(F_{k,1}^{R}+F_{k,2}^{R}+\cdots+F_{k,2n-1}^{R}+F_{k,2n}^{R})$$
$$+k(F_{k,1}^{R}+F_{k,3}^{R}+F_{k,5}^{R}+\cdots+F_{k,2n-1}^{R})-F_{k,2n}^{R}-kF_{k,2n-1}^{R}$$

$$\therefore \quad (2-k)(F^R_{k,1}+F^R_{k,3}+F^R_{k,5}+\cdots+F^R_{k,2n-1}) = 1-(F^R_{k,2n}+kF^R_{k,2n-1})+(F^R_{k,1}+F^R_{k,2}+\cdots+F^R_{k,2n})$$

$$\therefore \quad (2-\mathrm{k})(F^R_{k,1}+F^R_{k,3}+F^R_{k,5}+\cdots+F^R_{k,2n-1}) = 1-F^R_{k,2n+1}+\frac{1}{k}(F^R_{k,2n+2}-1)$$

$$\therefore \quad F^R_{k,1}+F^R_{k,3}+F^R_{k,5}+\cdots+F^R_{k,2n-1} = \frac{1}{k(2-k)}(F^R_{k,2n+2}-kF^R_{k,2n+1}+k-1).$$

**Lemma 3.2.5** $\sum_{i=1}^{n} F^R_{k,2i} = \frac{1}{k(2-k)}(F^R_{k,2n+2}-k^2F^R_{k,2n}-1)$.

*Proof*: We have $F^R_{k,n} = F^R_{k,n-1}+kF^R_{k,n-2}, \quad n \geq 2$ . Replacing $n$ by 2, 4, 6, …, we get

$$F^R_{k,2} = F^R_{k,1}+kF^R_{k,0}$$

$$F^R_{k,4} = F^R_{k,3}+kF^R_{k,2}$$

$$F^R_{k,6} = F^R_{k,5}+kF^R_{k,4}$$

$$\vdots$$

$$F^R_{k,2n} = F^R_{k,2n-1}+kF^R_{k,2n-2}$$

Adding all these equations term by term and using Lemma 3.2.2, we get

$$F^R_{k,2}+F^R_{k,4}+F^R_{k,6}+\cdots+F^R_{k,2n} = (F^R_{k,1}+F^R_{k,3}+F^R_{k,5}+\cdots+F^R_{k,2n-1}) + k(F^R_{k,2}+F^R_{k,4}+F^R_{k,6}+\cdots+F^R_{k,2n-2})$$

$$\therefore \quad 2(F^R_{k,2}+F^R_{k,4}+F^R_{k,6}+\cdots+F^R_{k,2n}) = (F^R_{k,1}+F^R_{k,2}+F^R_{k,3}+\cdots+F^R_{k,2n}) + k(F^R_{k,2}+F^R_{k,4}+F^R_{k,6}+\cdots+F^R_{k,2n})-kF^R_{k,2n}$$

$$\therefore \quad (2-k)(F^R_{k,2}+F^R_{k,4}+F^R_{k,6}+\cdots+F^R_{k,2n}) = (F^R_{k,1}+F^R_{k,2}+F^R_{k,3}+\cdots+F^R_{k,2n})-kF^R_{k,2n}$$

$$= \frac{1}{k}(F^R_{k,2n+2}-1)-kF^R_{k,2n}$$

$$\therefore \quad F^R_{k,2}+F^R_{k,4}+F^R_{k,6}+\cdots+F^R_{k,2n} = \frac{1}{k(2-k)}(F^R_{k,2n+2}-k^2F^R_{k,2n}-1).$$

We now prove a result which gives the value of product of two consecutive generalized Fibonacci numbers.

## 3.3 Some more identities for right $k$-Fibonacci numbers:

We now derive some more interesting identities for $F^R_{k,\mathrm{n}}$. First we prove an interesting reduction formula.

**Lemma 3.3.1** $F^R_{k,\mathrm{m+n}} = kF^R_{k,\mathrm{m}-1}F^R_{k,n} + F^R_{k,\mathrm{m}}F^R_{k,n+1}$.

*Proof*: Let *m* be the fixed positive integer. We proceed by inducting on *n*.

For $n=1$, we have $F^R_{k,\mathrm{m}+1} = kF^R_{k,\mathrm{m}-1}F^R_{k,1} + F^R_{k,\mathrm{m}}F^R_{k,2}$.

Since $F^R_{k,1} = F^R_{k,2} = 1$, we have $F^R_{k,m+1} = F^R_{k,m} + kF^R_{k,m-1}$, which is true.

This proves the result for $n=1$.

Now let us assume that the result is true for all integers up to some integer '*t*'.

Thus both $F^R_{k,\mathrm{m}+t} = kF^R_{k,\mathrm{m}-1}F^R_{k,t} + F^R_{k,\mathrm{m}}F^R_{k,t+1}$ and $F^R_{k,\mathrm{m}+(t-1)} = kF^R_{k,\mathrm{m}-1}F^R_{k,t-1} + F^R_{k,\mathrm{m}}F^R_{k,t}$ holds.

Now, from these two results we have $F^R_{k,\mathrm{m}+t} + kF^R_{k,m+(t-1)} = kF^R_{k,\mathrm{m}-1}(F^R_{k,t} + kF^R_{k,t-1}) + F^R_{k,m}(F^R_{k,t+1} + kF^R_{k,t})$

$$= kF^R_{k,\mathrm{m}-1}F^R_{k,t+1} + F^R_{k,\mathrm{m}}F^R_{k,t+2}$$

$$= kF^R_{k,\mathrm{m}-1}F^R_{k,t+1} + F^R_{k,\mathrm{m}}F^R_{k,(t+1)+1} = F^R_{k,\mathrm{m}+(t+1)}$$, this is obviously true.

Thus by the principal of mathematical induction, the result is true for all positive integers *n*.

It is often useful to extend the sequence of *right k-Fibonacci numbers* backward with negative subscripts. In fact, if we try to extend the *right k- Fibonacci sequence* backwards still keeping to the same rule, we get the following:

| $n$ | $F^R_{k,\mathrm{n}}$ |
|---|---|
| -1 | $\frac{1}{k}$ |
| -2 | $-\frac{1}{k^2}$ |
| -3 | $\frac{1+k}{k^3}$ |
| -4 | $-\frac{1+2k}{k^4}$ |
| -5 | $\frac{1+3k+k^2}{k^5}$ |
| -6 | $-\frac{1+4\mathrm{k}+3\mathrm{k}^2}{k^6}$ |
| -7 | $\frac{1+5k+6k^2+k^3}{k^7}$ |
| -8 | $-\frac{1+6\mathrm{k}+10\mathrm{k}^2+4\mathrm{k}^3}{k^8}$ |

Thus the sequence of *right k- Fibonacci numbers* is a bilateral sequence, since it can be extended infinitely in both directions. From this table and from the table of values of $F_{k,\mathrm{n}}^{R}$, the following result follows immediately:

**Lemma 3.3.2** $F_{k,-n}^{R} = \frac{(-1)^{n+1}}{k^{n}} F_{k,n}^{R},\ n \ge 1$.

**Note:** This result will prove later on by Binet's Formula.

We now obtain the extended d'Ocagne's Identity for this sequence.

**Lemma 3.3.3** $F_{k,m-n}^{R} = \frac{(-1)^{n}}{k^{n}} \left( F_{k,m}^{R} F_{k,n+1}^{R} - F_{k,m+1}^{R} F_{k,n}^{R} \right)$.

*Proof*: Replacing $n$ by $-n$ in Lemma 3.3.1, we get $F_{k,m-n}^{R} = kF_{k,m-1}^{R} F_{k,-n}^{R} + F_{k,m}^{R} F_{k,-n+1}^{R}$.

Using the definition of *left k- Fibonacci sequence* and Lemma 3.3.2,

we get

$$F_{k,m-n}^{R} = kF_{k,m-1}^{R} \frac{(-1)^{n+1}}{k^{n}} F_{k,n}^{R} + F_{k,m}^{R} \frac{(-1)^{n}}{k^{n-1}} F_{k,n-1}^{R}$$

$$= \frac{(-1)^{n}}{k^{n-1}} (F_{k,m}^{R} F_{k,n-1}^{R} - F_{k,m-1}^{R} F_{k,n}^{R})$$

$$= \frac{(-1)^{n}}{k^{n-1}} [F_{k,m}^{R} \frac{1}{k} (F_{k,n+1}^{R} - F_{k,n}^{R}) - \frac{1}{k} (F_{k,m+1}^{R} - F_{k,m}^{R}) F_{k,n}^{R}]$$

$$\therefore \quad F_{k,m-n}^{R} = \frac{(-1)^{n}}{k^{n}} \left( F_{k,m}^{R} F_{k,n+1}^{R} - F_{k,m+1}^{R} F_{k,n}^{R} \right).$$

We next prove the divisibility property for $F_{k,\mathrm{n}}^{R}$.

**Lemma 3.3.4** $F_{k,m}^{R} \mid F_{k,mn}^{R}$; for any non-zero integers $m$ and $n$.

*Proof*: Let $m$ be any fixed positive integer. We proceed by inducting on $n$.

For $n = 1$, we have $F_{k,m}^{R} \mid F_{k,m}^{R}$, which is obvious. This proves the result for $n = 1$.

Now assume that the result is true for all integers $n$ up to some integer '$t$'.

Thus $F_{k,m}^{R} \mid F_{k,mt}^{R}$ hold by assumption.

Then $F_{k,m(t+1)}^{R} = F_{k,mt+m}^{R} = kF_{k,mt-1}^{R} F_{k,m}^{R} + F_{k,mt}^{R} F_{k,m+1}^{R}$.

But by assumption, we have $F_{k,m}^{R} \mid F_{k,mt}^{R}$. Thus $F_{k,m}^{R}$ divides the entire right side of the above equation. Hence $F_{k,m}^{R} \mid F_{k,m(t+1)}^{R}$, this proves the result for all positive integers $n$.

**Note:** By Lemma 3.3.2 it is obvious that the above divisibility criterion holds for negative values of $n$ also.

**Lemma 3.3.5** $F_{k,n}^{R\,2}+\frac{1}{k}F_{k,n+1}^{R\;\;2}=\frac{1}{k}F_{k,2n+1}^{R}$.

*Proof*: Here also we use the principal of mathematical induction.

For $n=1$, we have $F_{k,1}^{R\,2}+\frac{1}{k}F_{k,2}^{R\,2}=1+\frac{1}{k}=\frac{1}{k}(1+\mathrm{k})=\frac{1}{k}\mathrm{F}_{k,3}^{R}$.

This proves the result for $n=1$.

We assume that it is true for all integers up to some positive integer '$t$'.

$\therefore \quad F_{k,t}^{R\,2}+\frac{1}{k}F_{k,t+1}^{R\;\;2}=\frac{1}{k}\mathrm{F}_{k,2t+1}^{R}$ holds by assumption.

Now $F_{k,t+1}^{R\;\;2}+\frac{1}{k}F_{k,t+2}^{R\;\;2}=\mathrm{F}_{k,t+1}^{R\;\;2}+\frac{1}{k}(\mathrm{F}_{k,t+1}^{R}+k\,\mathrm{F}_{k,t}^{R})^{2}$

$$=F_{k,t+1}^{R\;\;2}+\frac{1}{k}(\mathrm{F}_{k,t+1}^{R\;\;2}+2k\,\mathrm{F}_{k,t}^{R}\mathrm{F}_{k,t+1}^{R}+k^{2}F_{k,t}^{R\,2})$$

$$=F_{k,t+1}^{R\;\;2}+kF_{k,\mathrm{t}}^{R\,2}+\frac{1}{k}(\mathrm{F}_{k,t+1}^{R\;\;2}+k\,\mathrm{F}_{k,t}^{R}\mathrm{F}_{k,t+1}^{R}+kF_{k,t}^{R}\,\mathrm{F}_{k,t+1}^{R})$$

$$=k(\mathrm{F}_{k,t}^{R\,2}+\frac{1}{k}F_{k,t+1}^{R\;\;2})+\frac{1}{k}[F_{k,t+1}^{R}(F_{k,t+1}^{R}+kF_{k,t}^{R})+kF_{k,t}^{R}F_{k,t+1}^{R}]$$

$$=k(\frac{1}{k}F_{k,2t+1}^{R})+\frac{1}{k}[\mathrm{F}_{k,t+1}^{L}\mathrm{F}_{k,t+2}^{L}+kF_{k,t}^{L}F_{k,t+1}^{L}]$$

$$=\mathrm{F}_{k,2t+1}^{R}+\frac{1}{k}(\mathrm{k}\,F_{k,t}^{R}\,\mathrm{F}_{k,t+1}^{R}+F_{k,t+1}^{R}F_{k,t+2}^{R})$$

$$=\mathrm{F}_{k,2t+1}^{R}+\frac{1}{k}F_{k,t+1+t+1}^{R}=\frac{1}{k}F_{k,2t+3}^{R}=\mathrm{F}_{k,2(t+1)+1}^{R}$$

This proves the result by induction.

Now, we derive a result which connects three consecutive *right k- Fibonacci numbers* with odd subscript.

**Lemma 3.3.6** $F_{k,2n+5}^{R}-(2k+1)F_{k,2n+3}^{R}+k^{2}F_{k,2n+1}^{R}=0$.

*Proof*: By definition

$$F_{k,2n+5}^{R}=F_{k,2n+4}^{R}+kF_{k,2n+3}^{R}=(F_{k,2n+3}^{R}+kF_{k,2n+2}^{R})+kF_{k,2n+3}^{R}$$

$$=(k+1)F_{k,2n+3}^{R}+kF_{k,2n+2}^{R}.$$

Now $F^R_{k,2n+5} - (2k+1)F^R_{k,2n+3} + k^2 F^R_{k,2n+1}$

$$= (k+1)F^R_{k,2n+3} + kF^R_{k,2n+2} - (2k+1)F^R_{k,2n+3} + k^2 F^R_{k,2n+1}$$

$$= (k+1)F^R_{k,2n+3} - (2k+1)F^R_{k,2n+3} + k(F^R_{k,2n+2} + kF^R_{k,2n+1})$$

$$= (k+1)F^R_{k,2n+3} - (2k+1)F^R_{k,2n+3} + kF^R_{k,2n+3} = 0 .$$

We finally prove the analogous of one of the oldest identities involving the Fibonacci numbers - Cassini's identity.

**Lemma 3.3.7** $F^R_{k,n+1}.F^R_{k,n-1} - {F^R_{k,n}}^2 = (-1)^n.k^{n-1}$ .

*Proof*: We have $F^R_{k,n+1}.F^R_{k,n-1} - {F^R_{k,n}}^2 = (F^R_{k,n} + kF^R_{k,n-1})F^R_{k,n-1} - {F^R_{k,n}}^2$

$$= F^R_{k,n}F^R_{k,n-1} - {F^R_{k,n}}^2 + k{F^R_{k,n-1}}^2$$

$$= F^R_{k,n}(F^R_{k,n-1} - F^R_{k,n}) + k{F^R_{k,n-1}}^2$$

$$= F^R_{k,n}(-kF^R_{k,n-2}) + k{F^R_{k,n-1}}^2$$

$$= -k(F^R_{k,n}F^R_{k,n-2} - {F^R_{k,n-1}}^2)$$

Repeating the same process successively for right side, we get

$$F^R_{k,n+1}F^R_{k,n-1} - {F^R_{k,n}}^2 = (-k)^2(F^R_{k,n-1}F^R_{k,n-3} - {F^R_{k,n-2}}^2)$$

$$= (-k)^n(F^R_{k,1}.F^R_{k,-1} - {F^R_{k,0}}^2)$$

$$= (-k)^n(1.\frac{1}{k} - 0) = (-1)^n.k^{n-1}$$

Since the value of $F^R_{k,1} = 1, F^R_{k,0} = 0, F^R_{k,-1} = \frac{1}{k}$ , we have

$$\therefore \quad F^R_{k,n+1}.F^R_{k,n-1} - {F^R_{k,n}}^2 = (-1)^n.k^{n-1}$$

We now prove an interesting result which expresses the *right k– Fibonacci number* as the sum of the preceding *right k– Fibonacci numbers*.

**Lemma 3.3.8** $F^R_{k,n} = 1 + k\sum_{i=0}^{n-2} F^R_{k,i};\ n \geq 2$ .

*Proof:* We prove this result by the principal of mathematical induction.

For $n = 2$ , we have $LHS = F^R_{k,2} = 1 = 1 + k(0) = 1 + F^R_{k,0} = RHS$ .

This proves the result for $n=2$.

We assume that it is true for all integers up to some positive integer '$t$.

Thus $F_{k,t}^{R}=1+k\sum_{i=0}^{t-2}F_{k,i}^{R};\ t\geq 2$, holds by assumption.

Now we consider the right side of the result to be proved for $n=t+1$.

Since $F_{k,0}^{R}=0$,

$$RHS=1+k(\mathrm{F}_{k,1}^{R}+\mathrm{F}_{k,2}^{R}+\mathrm{F}_{k,3}^{R}\cdots+\mathrm{F}_{k,t-1}^{R})$$

$$=\mathrm{F}_{k,2}^{R}+k\,\mathrm{F}_{k,1}^{R}+k(\mathrm{F}_{k,2}^{R}+\mathrm{F}_{k,3}^{R}\cdots+\mathrm{F}_{k,t-1}^{R})$$

$$=F_{k,3}^{R}+kF_{k,2}^{R}+k(F_{k,3}^{R}+F_{k,4}^{R}+\cdots+F_{k,t-1}^{R})$$

Continuing this process on right hand side, the last term will be

$$\therefore\ RHS=F_{k,t}^{R}+kF_{k,t-1}^{R}=F_{k,t+1}^{R}=LHS$$

$\therefore$ The result is true for $n=t+1$. This proves the result by induction.

**Lemma 3.3.9** $\gcd(F_{k,n}^{R},F_{k,n+1}^{R})=1,\ \forall\ n=0,1,2,3,...$

*Proof*: Suppose that $F_{k,n}^{R}$ and $F_{k,n+1}^{R}$ are both divisible by a positive integer $d$. Then clearly

$$F_{k,n+1}^{R}-F_{k,n}^{R}=F_{k,n}^{R}+kF_{k,n-1}^{R}-F_{k,n}^{R}=kF_{k,n-1}^{R}$$

will also be divisible by $d$. Then right hand side of this result is divisible by $d$.

This gives $d\,/\,kF_{k,n-1}^{R}$.

First we claim that $F_{k,n}^{R}$ is always relatively prime to $k$.

From the lemma 3.3.8, we have $F_{k,n}^{R}=1+k\sum_{i=0}^{n-2}F_{k,i}^{R};\ n\geq 2$.

If some integer $d'>1$ is divisor of both $F_{k,n}^{R}$ and $k$, then from above result it is clear that $d'$ must divide 1, a contradiction. Thus $F_{k,n}^{R}$ is always relatively prime to $k$. $\Rightarrow d\mid F_{k,\mathrm{n}-1}^{R}$.

Continuing this argument we see that $d\mid F_{k,n-2}^{R}$, $d\mid F_{k,n-3}^{R}$ and so on. Eventually, we must have $d\mid F_{k,1}^{R}$. Since $F_{k,1}^{R}=1$ we get $d=1$, this proves the required result.

As we define matrix in Chapter- 2, we demonstrate a close link between matrices and *right k- Fibonacci numbers*. We define an important $2\times 2$ matrix $U=\begin{bmatrix}0 & k\\ 1 & 1\end{bmatrix}$, which plays a significant role in discussions concerning *right k- Fibonacci sequence*.

We prove the following results, which will be used later in the chapter.

**Lemma 3.3.10** (1) $U^2 = U + kI$ (2) $U^{-2} = \frac{1}{k}\left(I - U^{-1}\right)$.

*Proof:* We have $U^2 = \begin{bmatrix} 0 & k \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & k \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} k & k \\ 1 & 1+k \end{bmatrix}$

$$= \begin{bmatrix} 0 & k \\ 1 & 1 \end{bmatrix} + k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= U + kI$$

(2) Again $U^2 = \begin{bmatrix} k & k \\ 1 & 1+k \end{bmatrix} \Rightarrow U^{-2} = \begin{bmatrix} k & k \\ 1 & 1+k \end{bmatrix}^{-1} = \frac{1}{k^2}\begin{bmatrix} 1+k & -k \\ -1 & k \end{bmatrix}$

and $U^{-1} = \begin{bmatrix} 0 & k \\ 1 & 1 \end{bmatrix}^{-1} = -\frac{1}{k}\begin{bmatrix} 1 & -k \\ -1 & 0 \end{bmatrix}$

$\therefore \frac{1}{k}\left(I - U^{-1}\right) = \frac{1}{k}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{k^2}\begin{bmatrix} 1 & -k \\ -1 & 0 \end{bmatrix}$

$$= \frac{1}{k^2}\left(\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} + \begin{bmatrix} 1 & -k \\ -1 & 0 \end{bmatrix}\right)$$

$= \frac{1}{k^2}\begin{bmatrix} k+1 & -k \\ -1 & k \end{bmatrix} = U^{-2}$, this proves the results.

**Lemma 3.3.11** If $U = \begin{bmatrix} 0 & k \\ 1 & 1 \end{bmatrix}$ then $U^n = \begin{bmatrix} kF^R_{k,n-1} & kF^R_{k,n} \\ F^R_{k,n} & F^R_{k,n+1} \end{bmatrix}$.

*Proof:* We will prove this result by using principal of mathematical induction.

For $n = 1$, we have $U = \begin{bmatrix} kF^R_{k,0} & kF^R_{k,1} \\ F^R_{k,1} & F^R_{k,2} \end{bmatrix} = \begin{bmatrix} 0 & k \\ 1 & 1 \end{bmatrix}$, this proves the result for $n = 1$.

Assume that it is true for $n = t$. Thus $U^t = \begin{bmatrix} kF^R_{k,t-1} & kF^R_{k,t} \\ F^R_{k,t} & F^R_{k,t+1} \end{bmatrix}$ holds.

Now $U^{t+1} = U^tU = \begin{bmatrix} kF^R_{k,t-1} & kF^R_{k,t} \\ F^R_{k,t} & F^R_{k,t+1} \end{bmatrix}\begin{bmatrix} 0 & k \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} kF^R_{k,t} & kF^R_{k,t} + k^2F^R_{k,t-1} \\ F^R_{k,t+1} & F^R_{k,t+1} + kF^R_{k,t} \end{bmatrix}$

$$= \begin{bmatrix} kF^R_{k,t} & kF^R_{k,t+1} \\ F^R_{k,t+1} & F^R_{k,t+2} \end{bmatrix}$$

Thus the result is true for $n = t+1$ also. This proves the result by induction.

**Remark:** Lemma: 3.3.7 can also be proved by using this matrix U.

We have, $U = \begin{bmatrix} 0 & k \\ 1 & 1 \end{bmatrix}$ then $|U| = -k$.

Also $U^n = \begin{bmatrix} kF^R_{k,n-1} & kF^R_{k,n} \\ F^R_{k,n} & F^R_{k,n+1} \end{bmatrix}$. Then $|U^n| = kF^R_{k,n-1}F^R_{k,n+1} - kF^{R\ 2}_{k,n}$.

$\therefore\ |U^n| = k\left[F^R_{k,n-1}F^R_{k,n+1} - F^{R\ 2}_{k,n}\right]$

$\therefore\ (-k)^n = k\left[F^R_{k,n-1}F^R_{k,n+1} - F^{R\ 2}_{k,n}\right]$

$\therefore\ F^R_{k,n-1}F^R_{k,n+1} - F^{R\ 2}_{k,n} = (-1)^n k^{n-1}$.

We can apply above Lemma to derive four new identities as the next corollary shows, although they are basically the same.

In the next article, first, we describe the terms of the *right k-Fibonacci sequence* $F^R_{k,n}$ explicitly by using a generalization of *Binet's formula*. We will start the main content of the second part by deriving a generalization of *Binet's formula* (Via generating functions) and then will present extensions of well known Fibonacci Identities such as Cassini's, Catalan's, d'Ocagne's.

## 3.4 Generating function for right k-Fibonacci number:

Generating functions provided a powerful technique for solving linear homogeneous recurrence relations. In this section, we consider the generating functions for the generalized *right k- Fibonacci sequence* and derive some identities satisfied by this sequence.

**Lemma 3.4.1** The generating function for the generalized *right k -Fibonacci sequence* $\{F^R_{k,n}\}$ is given by $f(x) = \dfrac{x}{1-x-kx^2}$.

*Proof*: We begin with the formal power series representation of generating function for $\{F^R_{k,n}\}$.

$$f(x) = \sum_{m=0}^{\infty} F^R_{k,m}x^m = \sum_{m=0}^{\infty} g_m x^m = g_0 + g_1x + g_2x^2 + g_3x^3 + \dots\ .$$

$$= 0 + (1)x + (g_1 + kg_0)x^2 + (g_2 + kg_1)x^3 + (g_3 + kg_2)x^4 + \cdots$$

$$= x + (g_0 + g_1 x + g_2 x^2 + \cdots)x + kx^2(g_1 x + g_2 x^2 + \cdots)$$

$$= x + xf(x) + kx^2 f(x)$$

$$\therefore \quad (1 - x - kx^2) f(x) = x \Rightarrow f(x) = \frac{x}{1 - x - kx^2}.$$

This is the generating function of $\left\{F_{k,n}^R\right\}$.

## 3.5 Extended Binet's formula for $F_{k,n}^R$:

For any positive real number $k$, the *right* $k$*-Fibonacci sequence* $\left\{F_{k,n}^R\right\}_{n\in N}$ is defined recurrently by $F_{k,n+1}^R = F_{k,n}^R + kF_{k,n-1}^R, n \ge 1$ where $F_{k,0}^R = 0, F_{k,1}^R = 1$ which is second order difference equation with constant coefficients. Therefore, it has the characteristic equation, $x^2 - x - k = 0$.

The roots of the characteristic equations are $\alpha = \frac{1+\sqrt{1+4k}}{2}$ and $\beta = \frac{1-\sqrt{1+4k}}{2}$. For $k > 0, \beta < 0 < \alpha, |\beta| < \alpha$. Also $\alpha + \beta = 1,\ \alpha\beta = -k$ and $\alpha - \beta = \sqrt{1+4k}$. The following theorem is the *extended Binet's formula for* $F_{k,n}^R$.

**Theorem 3.5.1** Prove that $F_{k,n}^R = \frac{\alpha^n - \beta^n}{\alpha - \beta}$.

*Proof:* The generating function for the generalized *right k-Fibonacci sequence* is given by $f(x) = \frac{x}{1 - x - kx^2}$. We rewrite $f(x)$ as $f(x) = \frac{x}{(1-\alpha x)(1-\beta x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$

Solving these we get $A = \frac{1}{\alpha - \beta},\ B = \frac{-1}{\alpha - \beta}$. Thus

$$f(x) = \frac{1}{\alpha-\beta}\left[\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right] = \frac{1}{\alpha-\beta}\left[(1-\alpha x)^{-1} - (1-\beta x)^{-1}\right]$$

$$= \frac{1}{\alpha-\beta}[(1+\alpha x+\alpha^2 x^2+\cdots) - (1+\beta x+\beta^2 x^2+\cdots)]$$

$$= \frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}(\alpha^n - \beta^n)x^n = \sum_{n=0}^{\infty}\frac{\alpha^n - \beta^n}{\alpha-\beta}x^n = \sum_{n=0}^{\infty} g_n x^n$$

Since we have $g_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$. Hence $F_{k,n}^L = \frac{\alpha^n - \beta^n}{\alpha - \beta}$.

This is the *extended Binet's formula* for $F_{k,n}^R$.

Particular cases of these sequences are as follows:

- If $k=1$, for the classical Fibonacci sequence, we have $\alpha=\frac{1+\sqrt{5}}{2}$ and $\beta=\frac{1-\sqrt{5}}{2}$, $\alpha$ is known as the golden ratio.
- If $k=2$, for the Pell's sequence, we have $\alpha=2$ and $\beta=-1$, $\alpha$ is known as the silver ratio.
- Finally, if $k=3$ for the sequence $\{H_n\}$, we have $\alpha=\frac{1+\sqrt{13}}{2}$, $\beta=\frac{1-\sqrt{13}}{2}$ where $\alpha$ is known as bronze ratio.

**Remark:** We prove the Lemma 3.3.2 by using the *Binet's formula*.

We have $F_{k,n}^{R}=\frac{\alpha^n-\beta^n}{\alpha-\beta}$. Substituting $-n$ in place of $n$, we get

$F_{k,-n}^{R}=\frac{\alpha^{-n}-\beta^{-n}}{\alpha-\beta}=\frac{-\left(\alpha^n-\beta^n\right)}{\alpha^n\beta^n\left(\alpha-\beta\right)}$. Hence $F_{k,-n}^{R}=\frac{(-1)^{n+1}}{k^n}F_{k,n}^{R}$, $n\geq 1$, as required.

**Lemma 3.5.2** (Extended Catalan's identity) $F_{k,n-r}^{R}F_{k,n+r}^{R}-F_{k,n}^{R\,2}=(-1)^{n+1-r}k^{n-r}F_{k,r}^{R\,2}$

*Proof:* By using theorem 3.5.1 and $\alpha+\beta=1$, $\alpha\beta=-k$ in LHS, we get

$$LHS=\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta}\frac{\alpha^{n+r}-\beta^{n+r}}{\alpha-\beta}-\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)^2$$

$$=\frac{\left(\alpha^{n-r}-\beta^{n-r}\right)\left(\alpha^{n+r}-\beta^{n+r}\right)-\alpha^{2n}+2\alpha^n\beta^n-\beta^{2n}}{\left(\alpha-\beta\right)^2}$$

$$=\frac{1}{\left(\alpha-\beta\right)^2}\left[-\left(\alpha\beta\right)^{n-r}\beta^{2r}-\left(\alpha\beta\right)^{n-r}\alpha^{2r}+2\left(\alpha\beta\right)^n\right]$$

$$=\frac{-\left(\alpha\beta\right)^{n-r}}{\left(\alpha-\beta\right)^2}\left[\beta^{2r}+\alpha^{2r}-2\left(\alpha\beta\right)^r\right]=\frac{-\left(-k\right)^{n-r}}{\left(\alpha-\beta\right)^2}\left(\alpha^r-\beta^r\right)^2$$

$$=\left(-1\right)^{n-r+1}k^{n-r}\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)^2=\left(-1\right)^{n-r+1}k^{n-r}F_{k,r}^{R\,2}.$$

This proves the required result.

Now we derive the extended d'Ocagne's identity for $F_{k,n}^{R}$.

**Lemma 3.5.3** If $m>n$ then $F_{k,\mathrm{m}}^{R}F_{k,n+1}^{R}-F_{k,\mathrm{m}+1}^{R}F_{k,n}^{R}=(-1)^n k^n F_{k,\mathrm{m-n}}^{R}$.

*Proof*: By using theorem 3.5.1 and $\alpha+\beta=1,\alpha\beta=-k$ on LHS, we get

$$LHS=\frac{\alpha^m-\beta^m}{\alpha-\beta}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}-\frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta}\frac{\alpha^n-\beta^n}{\alpha-\beta}$$

$$=\frac{1}{(\alpha-\beta)^2}\left[\left(\alpha^m-\beta^m\right)\left(\alpha^{n+1}-\beta^{n+1}\right)-\left(\alpha^{m+1}-\beta^{m+1}\right)\left(\alpha^n-\beta^n\right)\right]$$

$$=\frac{(\alpha\beta)^n}{(\alpha-\beta)^2}\left[-\frac{\alpha^m\beta^{n+1}}{(\alpha\beta)^n}-\frac{\alpha^{n+1}\beta^m}{(\alpha\beta)^n}+\frac{\alpha^{m+1}\beta^n}{(\alpha\beta)^n}+\frac{\alpha^n\beta^{m+1}}{(\alpha\beta)^n}\right]$$

$$=\frac{(-k)^n}{(\alpha-\beta)^2}\left[-\alpha^{m-n}\beta-\alpha\beta^{m-n}+\alpha^{m-n+1}+\beta^{m-n+1}\right]$$

$$=\frac{(-k)^n}{(\alpha-\beta)^2}\left[\alpha^{m-n}(\alpha-\beta)-\beta^{m-n}(\alpha-\beta)\right]$$

$$=\frac{(-k)^n}{(\alpha-\beta)^2}\left(\alpha^{m-n}-\beta^{m-n}\right)(\alpha-\beta).$$

**Lemma 3.5.4** The sum of first $(n+1)$ terms of the *right k- Fibonacci numbers* is given by $\sum_{i=0}^{n}F_{k,i}^R=S_{k,n}^R=\frac{1}{k}\left(F_{k,n+2}^R-1\right)$.

*Proof*: By theorem 3.5.1, $S_{k,n}^R$ can be written as

$$S_{k,n}^R=\frac{1}{\alpha-\beta}\sum_{i=0}^{n}(\alpha^i-\beta^i)=\frac{1}{\alpha-\beta}\left[\sum_{i=0}^{n}\alpha^i-\sum_{i=0}^{n}\beta^i\right]$$

$$=\frac{1}{\alpha-\beta}\left(\frac{\alpha^{n+1}-1}{\alpha-1}-\frac{\beta^{n+1}-1}{\beta-1}\right)$$

$$=\frac{\alpha^{n+1}\beta-\beta-\alpha^{n+1}+1-\alpha\beta^{n+1}+\alpha+\beta^{n+1}-1}{(\alpha-\beta)(\alpha-1)(\beta-1)}$$

$$=\frac{-k\alpha^n+k\beta^n+\alpha-\beta-\alpha^{n+1}+\beta^{n+1}}{(\alpha-\beta)(-k)}$$

$$=\frac{-k\left(\alpha^n-\beta^n\right)+(\alpha-\beta)-\left(\alpha^{n+1}-\beta^{n+1}\right)}{(\alpha-\beta)(-k)}$$

Since, we have $(\alpha-1)(\beta-1)=-k$ and $\alpha=\frac{1+\sqrt{1+4k}}{2}$, $\beta=\frac{1-\sqrt{1+4k}}{2}$ we get

$$S_{k,n}^{R}=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}+\frac{1}{k}\left[\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}-1\right]=\frac{1}{k}\left[F_{k,n+1}^{R}+kF_{k,n}^{R}-1\right]$$

$$\therefore S_{k,n}^{R}=\frac{1}{k}\left[F_{k,n+2}^{R}-1\right].$$

**Lemma 3.5.5** $\lim\limits_{x\to\infty}\frac{F_{k,n}^{R}}{F_{k,n-1}^{R}}=\alpha$.

*Proof:* By using theorem 3.5.1, we have

$$\lim_{x\to\infty}\frac{F_{k,n}^{R}}{F_{k,n-1}^{R}}=\lim_{x\to\infty}\frac{\alpha^{n}-\beta^{n}}{\alpha^{n-1}-\beta^{n-1}}=\alpha\lim_{x\to\infty}\frac{1-\left(\frac{\beta}{\alpha}\right)^{n}}{1-\left(\frac{\beta}{\alpha}\right)^{n-1}}=\alpha$$

This happens since, we have $\lim\limits_{x\to\infty}\left(\frac{\beta}{\alpha}\right)^{n}=0,\quad |\beta|<\alpha$.

We finally prove the following combinatorial identity for $F_{k,n}^{R}$.

**Lemma 3.5.6** $F_{k,n}^{R}=\frac{1}{2^{n-1}}\sum\limits_{i=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2i+1}(1+4k)^{i}$.

*Proof:* By using $\alpha=\frac{1+\sqrt{1+4k}}{2}$, $\beta=\frac{1-\sqrt{1+4k}}{2}$ and $\alpha-\beta=\sqrt{1+4k}$, $\alpha\beta=-k$

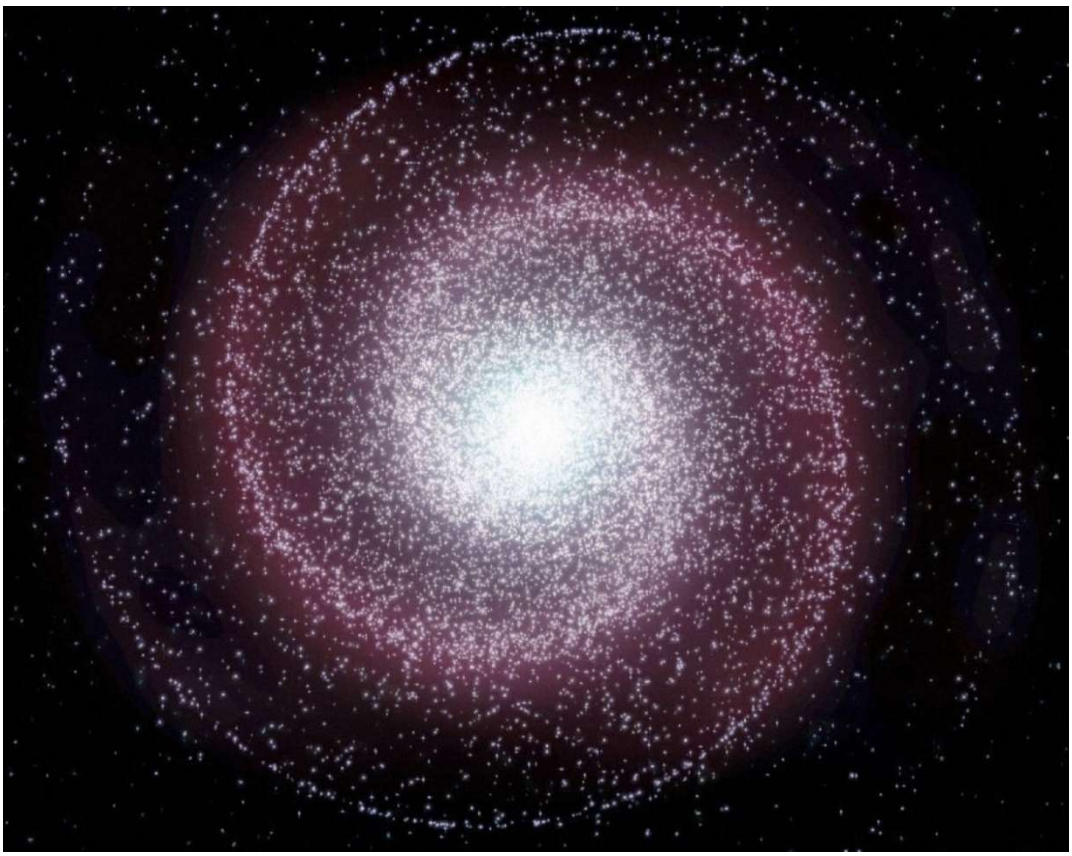
$$F_{k,n}^{R}=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}=\frac{1}{\sqrt{1+4k}}\left[\left(\frac{1+\sqrt{1+4k}}{2}\right)^{n}-\left(\frac{1-\sqrt{1+4k}}{2}\right)^{n}\right]$$

$$=\frac{1}{2^{n}\sqrt{1+4k}}\left[\left(1+\sqrt{1+4k}\right)^{n}-\left(1-\sqrt{1+4k}\right)^{n}\right]$$

$$=\frac{1}{2^{n}\sqrt{1+4k}}\left\{\begin{array}{l}\left[\binom{n}{1}\sqrt{1+4k}+\binom{n}{2}\left(\sqrt{1+4k}\right)^{2}+\cdots\right]\\-\left[-\binom{n}{1}\sqrt{1+4k}+\binom{n}{2}\left(\sqrt{1+4k}\right)^{2}-\cdots\right]\end{array}\right\}$$

$$=\frac{1}{2^{n}\sqrt{1+4k}}\left[2\binom{n}{1}\sqrt{1+4k}+2\binom{n}{3}\left(\sqrt{1+4k}\right)^{3}+\cdots\right]\quad\therefore\quad F_{k,n}^{R}=\frac{1}{2^{n-1}}\sum_{i=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2i+1}(1+4k)^{i}.$$

# Chapter – 4



# Associated left and right k- Fibonacci numbers

## 4.1 Introduction:

In [1] Alvaro H. Salas defined the sequence $\left\{A_{k,n}\right\}_{n=0}^{\infty}$ associated to $\left\{F_{k,n}\right\}_{n=0}^{\infty}$ as $A_{k,n} = F_{k,n} + F_{k,n-1}$ where $A_{k,0} = 1$ for $n$=1,2,3,…. He then defines associated $k$- Fibonacci numbers by recurrence relation

$$A_{k,n} = kA_{k,n-1} + A_{k,n-2};\ n \geq 2 \text{ where } A_{k,0} = 1.$$

In this chapter we define *associated left k- Fibonacci sequence* $\left\{A_{k,n}^{L}\right\}$ and *associated right k- Fibonacci sequence* $\left\{A_{k,n}^{R}\right\}$ and prove some interesting properties of both *associated left k- Fibonacci numbers* and *associated right k- Fibonacci numbers*. Despite its simple appearance, this sequence contains a wealth of subtle and fascinating properties. In this chapter we explore several of the fundamental identities related with $A_{k,n}^{L}$ and $A_{k,n}^{R}$.

## 4.2 The associated left *k*-Fibonacci numbers:

**Definition:** The sequence of *associated left k- Fibonacci numbers* $\left\{A_{k,n}^{L}\right\}$ associate to left k-Fibonacci sequence $\left\{F_{k,n}^{L}\right\}$ is defined as $A_{k,0}^{L} = 1$ and $A_{k,n}^{L} = F_{k,n}^{L} + F_{k,n-1}^{L}, n = 1,2,3\ldots.$

We observe that the expression $A_{k,n}^{L}$ is the sum of the two consecutive *left k-Fibonacci numbers* $F_{k,n}^{L}$ and its predecessor $F_{k,n-1}^{L}$. The members of the sequence $\left\{A_{k,n}^{L}\right\}$ will be called *associated left k-Fibonacci numbers*. An equivalent definition for the sequence $\left\{A_{k,n}^{L}\right\}$ is

$$A_{k,n}^{L} = \begin{cases} 1 & ,if\ n = 0 \\ 1 & ,if\ n = 1 \\ (k+1)F^{L}{}_{k,n-1} + F^{L}{}_{k,n-2} & ,if\ n \geq 2 \end{cases}$$

Observe that $A_{k,n}^{L} = F_{k,n}^{L} + F_{k,n-1}^{L} = kF_{k,n-1}^{L} + F_{k,n-2}^{L} + kF_{k,n-2}^{L} + F_{k,n-3}^{L}$

$$= k(F_{k,n-1}^{L} + F_{k,n-2}^{L}) + F_{k,n-2}^{L} + F_{k,n-3}^{L} = kA_{k,n-1}^{L} + A_{k,n-2}^{L}$$

This allows defining recursively the sequence of associated left $k$-Fibonacci numbers as follows:

$$A_{k,n}^{L} = \begin{cases} 1, if \;\; n = 0 \\ 1, if \;\; n = 1 \\ kA_{k,n-1}^{L} + A_{k,n-2}^{L}, if \;\; n \geq 2 \end{cases}$$

Members of *associated left k- Fibonacci sequence* $\left\{A_{k,n}^{L}\right\}$ will be called *associated left k-Fibonacci numbers*. Some of them are

| $n$ | $A_{k,n}^{L}$ |
|---|---|
| 0 | $1$ |
| 1 | $1$ |
| 2 | $k+1$ |
| 3 | $k^2+k+1$ |
| 4 | $k^3+k^2+2k+1$ |
| 5 | $k^4+k^3+3k^2+2k+1$ |
| 6 | $k^5+k^4+4k^3+3k^2+3k+1$ |
| 7 | $k^6+k^5+5k^4+4k^3+6k^2+3k+1$ |
| 8 | $k^7+k^6+6k^5+5k^4+10k^3+6k^2+4k+1$ |
| 9 | $k^8+k^7+7k^6+6k^5+15k^4+10k^3+10k^2+4k+1$ |
| 10 | $k^9+k^8+8k^7+7k^6+21k^5+15k^4+20k^3+10k^2+5k+1$ |

Appendix-III is computer program to obtain terms of *associated left k- Fibonacci sequence* $\left\{A_n\right\}$ using the programming language MATLAB (R2008a).

## 4.3 Basic identities of associated left *k*- Fibonacci numbers:

One of the purposes of this chapter is to develop many identities and results. We use the technique of induction as a useful tool in proving many of these identities and theorems involving Fibonacci numbers.

**Lemma 4.3.1** $\gcd(\mathrm{A}_{k,n}^{L}, \mathrm{A}_{k,n+1}^{L}) = 1, \;\; \forall \; n = 0,1,2,3,...$

*Proof:* Suppose that $A_{k,n}^{L}$ and $A_{k,n+1}^{L}$ are both divisible by a positive integer $d$. Then their difference $A_{k,n+1}^{L} - A_{k,n}^{L} = kA_{k,n}^{L} + A_{k,n-1}^{L} - A_{k,n}^{L} \;\; = (k-1)\,\mathrm{A}_{k,n}^{L} + A_{k,n-1}^{L}$

will also be divisible by $d$. Then right hand side of this equation is divisible by $d$. Thus $d \mid \mathrm{A}_{k,n-1}^{L}$. Continuing we see that $d \mid \mathrm{A}_{k,n-2}^{L}$, $d \mid \mathrm{A}_{k,n-3}^{L}$ and so on…

Eventually, we must have $d \mid \mathrm{A}_{k,1}^{L}$. But $A_{k,1}^{L} = 1$ then $d = 1$. Since the only positive integer which divides successive terms of the *associated left k - Fibonacci sequence* is 1. This proves the required result.

Next we derive the formula for sum of the first *n associated left k - Fibonacci numbers*.

**Lemma 4.3.2** $A_{k,1}^{L} + A_{k,2}^{L} + A_{k,3}^{L} + \cdots + A_{k,n}^{L} = \frac{1}{k}(\mathrm{A}_{k,n+1}^{L} + A_{k,n}^{L} - 2)$.

*Proof:* We have $A_{k,n}^{L} = kA_{k,n-1}^{L} + A_{k,n-2}^{L}, \quad n \geq 2$.

Replacing *n* by 2, 3, 4… we have

$$A_{k,2}^{L} = kA_{k,1}^{L} + A_{k,0}^{L},$$

$$A_{k,3}^{L} = kA_{k,2}^{L} + A_{k,1}^{L},$$

$$A_{k,4}^{L} = kA_{k,3}^{L} + A_{k,2}^{L}$$

$$\vdots$$

$$A_{k,n-1}^{L} = kA_{k,n-2}^{L} + A_{k,n-3}^{L}$$

$$A_{k,n}^{L} = kA_{k,n-1}^{L} + A_{k,n-2}^{L}$$

Adding all these equations term by term, we get

$$A_{k,1}^{L} + A_{k,2}^{L} + A_{k,3}^{L} + \cdots + A_{k,n}^{L}$$

$$= A_{k,1}^{L} + (k+1)(\mathrm{A}_{k,1}^{L} + A_{k,2}^{L} + \cdots + A_{k,\mathrm{n}-2}^{L}) + \mathrm{kA}_{k,\mathrm{n}-1}^{L} + \mathrm{A}_{k,0}^{L}$$

$$= A_{k,0}^{L} + A_{k,1}^{L} + (k+1)(\mathrm{A}_{k,1}^{L} + A_{k,2}^{L} + \cdots + A_{k,n}^{L}) - \mathrm{A}_{k,\mathrm{n}-1}^{L} - (\mathrm{k}+1)A_{k,\mathrm{n}}^{L}$$

$$\therefore \quad (1-\mathrm{k}-1)(\mathrm{A}_{k,1}^{L} + A_{k,2}^{L} + \cdots + A_{k,n}^{L}) = A_{k,0}^{L} + A_{k,1}^{L} - (A_{k,n-1}^{L} + kA_{k,n}^{L} + A_{k,n}^{L})$$

$$\therefore \quad -k(\mathrm{A}_{k,1}^{L} + A_{k,2}^{L} + \cdots + A_{k,n}^{L}) = 2 - (kA_{k,\mathrm{n}+1}^{L} + A_{k,n}^{L})$$

$$\therefore \; A_{k,1}^{L} + A_{k,2}^{L} + A_{k,3}^{L} + \cdots + A_{k,n}^{L} = \frac{1}{k}(\mathrm{A}_{k,n+1}^{L} + A_{k,n}^{L} - 2).$$

An alternate method of proving Lemma 4.3.2 is to apply the principle of mathematical induction. Using the same process or by induction we can derive formulae for the sum of the first *n associated left k-Fibonacci numbers* with various subscripts.

We next derive the sum of first *n associated left* $k$ *−Fibonacci numbers* with only odd or even subscripts.

**Lemma 4.3.3** $A_{k,1}^L + A_{k,3}^L + A_{k,5}^L + \cdots + A_{k,2n-1}^L = \frac{1}{k}(A_{k,2n}^L - 1)$.

*Proof:* We have $A_{k,n}^L = kA_{k,n-1}^L + A_{k,n-2}^L$, $n \geq 2$. Replacing $n$ by 2, 4, 6… we have

$$A_{k,2}^L = kA_{k,1}^L + A_{k,0}^L$$

$$A_{k,4}^L = kA_{k,3}^L + A_{k,2}^L$$

$$A_{k,6}^L = kA_{k,5}^L + A_{k,4}^L$$

$$\vdots$$

$$A_{k,2n-2}^L = kA_{k,2n-3}^L + A_{k,2n-4}^L$$

$$A_{k,2n}^L = kA_{k,2n-1}^L + A_{k,2n-2}^L$$

Adding all these equations term by term,

$$A_{k,2}^L + A_{k,4}^L + A_{k,6}^L + \cdots + A_{k,2n}^L$$

$$= k(A_{k,1}^L + A_{k,3}^L + \cdots + A_{k,2n-1}^L) + (A_{k,0}^L + A_{k,2}^L + A_{k,4}^L + \cdots + A_{k,2n-2}^L)$$

$$= A_{k,0}^L + k(A_{k,1}^L + A_{k,3}^L + \cdots + A_{k,2n-1}^L) + (A_{k,2}^L + A_{k,4}^L + \cdots + A_{k,2n}^L) - A_{k,2n}^L$$

$$\therefore \quad 0 = A_{k,0}^L + k(A_{k,1}^L + A_{k,3}^L + \cdots + A_{k,2n-1}^L) - A_{k,2n}^L$$

$$\therefore \quad A_{k,1}^L + A_{k,3}^L + \cdots + A_{k,2n-1}^L = \frac{1}{k}(A_{k,2n}^L - 1).$$

**Lemma 4.3.4** $A_{k,2}^L + A_{k,4}^L + A_{k,6}^L + \cdots + A_{k,2n}^L = \frac{1}{k}(A_{k,2n+1}^L - 1)$.

*Proof:* We know that $A_{k,n}^L = kA_{k,n-1}^L + A_{k,n-2}^L$, $n \geq 2$

Replacing $n$ by 1, 3, 5 … we have

$$A_{k,1}^L = 1$$

$$A_{k,3}^L = kA_{k,2}^L + A_{k,1}^L$$

$$A_{k,5}^L = kA_{k,4}^L + A_{k,3}^L$$

$$\vdots \quad \vdots \quad \vdots$$

$$A_{k,2n-1}^L = kA_{k,2n-2}^L + A_{k,2n-3}^L$$

Adding all these equations term by term,

$$A^L_{k,1}+A^L_{k,3}+A^L_{k,5}+\cdots+A^L_{k,2n-1}$$

$$=\mathrm{A}^L_{k,1}+k(\mathrm{A}^L_{k,2}+A^L_{k,4}+\cdots+A^L_{k,2n-2})+\left(A^L_{k,1}+A^L_{k,3}+\cdots+A^L_{k,2n-3}\right)$$

$$=\mathrm{A}^L_{k,1}+k(\mathrm{A}^L_{k,2}+A^L_{k,4}+\cdots+A^L_{k,2n})-\mathrm{kA}^L_{k,2n}+$$

$$(\mathrm{A}^L_{k,1}+A^L_{k,3}+\cdots+A^L_{k,2n-1})-A^L_{k,2n-1}$$

$$\therefore\quad 0=1+k(\mathrm{A}^L_{k,2}+A^L_{k,4}+\cdots+A^L_{k,2n})-(\mathrm{kA}^L_{k,2n}+A^L_{k,2n-1}).$$

$$\therefore\quad A^L_{k,2}+A^L_{k,4}+A^L_{k,6}+\cdots+A^L_{k,2n}=\frac{1}{k}(\mathrm{A}^L_{k,2n+1}-1).$$

The following results follow immediately from above two lemmas.

**Corollary 4.3.5** $A^L_{k,2n}\equiv 1(\bmod k)$ and $A^L_{k,2n+1}\equiv 1(\bmod \mathrm{k})$.

*Proof:* We use Mathematical Induction to prove the first result.

For $n=1$, we have $A^L_{k,2}=1+k\equiv 1\left(mod\ k\right)$

Suppose it is true for $n=r$, Thus $F^R_{k,2\mathrm{r}}\equiv 1\left(mod\ k\right)$ holds.

Now, $A^L_{k,2\mathrm{r}+2}=kA^L_{k,2\mathrm{r}+1}+A^L_{k,2\mathrm{r}}\equiv 1\left(mod\ k\right)$.

So the result is true for $n=r+1$ also. This proves the result for all integers *n*.

Similarly, we can prove the second result.

We next investigate the interesting new reduction formula for $A^L_{k,n}$.

**Lemma 4.3.6** $A^L_{k,\mathrm{m+n}}=A^L_{k,\mathrm{m}-1}A^L_{k,n}+A^L_{k,\mathrm{m}}A^L_{k,n+1}-\mathrm{A}^L_{k,\mathrm{m+n}-1}$.

*Proof* Let $m$ be a fixed integer and we proceed by inducting on $n$.

For $n=1$, we have

$$A^L_{k,\mathrm{m}+1}=A^L_{k,\mathrm{m}-1}A^L_{k,1}+A^L_{k,\mathrm{m}}A^L_{k,2}-\mathrm{A}^L_{k,\mathrm{m}}$$

$$A^L_{k,\mathrm{m}+1}=A^L_{k,\mathrm{m}-1}(1)+A^L_{k,\mathrm{m}}(\mathrm{k}+1)-\mathrm{A}^L_{k,\mathrm{m}}$$

$$A^L_{k,\mathrm{m}+1}=kA^L_{k,\mathrm{m}}+A^L_{k,\mathrm{m}-1},\quad (\because A^L_{k,1}=1,\ A^L_{k,2}=k+1)$$

This is obvious.

Now let us assume that the result is true for $n=1,2,3,\cdots,\mathrm{t}$;

$$A^L_{k,m+t} = A^L_{k,m-1}A^L_{k,t} + A^L_{k,m}A^L_{k,t+1} - A^L_{k,m+t-1} \text{ and}$$

$$A^L_{k,m+(t-1)} = A^L_{k,m-1}A^L_{k,t-1} + A^L_{k,m}A^L_{k,t} - A^L_{k,m+t-2}$$

We will show that it holds for $n = t+1$, also from above two equation, we have

$$\begin{aligned} A^L_{k,m+t+1} &= kA^L_{k,m+t} + A^L_{k,m+(t-1)} \\ &= k(A^L_{k,m-1}A^L_{k,t} + A^L_{k,m}A^L_{k,t+1}) + (A^L_{k,m-1}A^L_{k,t-1} + A^L_{k,m}A^L_{k,t}) \\ &= A^L_{k,m-1}(kA^L_{k,t} + A^L_{k,t-1}) + A^L_{k,m}(kA^L_{k,t+1} + A^L_{k,t}) - (k\,A^L_{k,m+t-1} + A^L_{k,m+t-2}) \\ &= A^L_{k,m-1}A^L_{k,t+1} + A^L_{k,m}A^L_{k,(t+1)+1} - A^L_{k,m+t} = A^L_{k,m+(t+1)} \end{aligned}$$

Thus the result is true for all $n \in N$. This proves the lemma.

It is often useful to extend the sequence of *associated left k- Fibonacci numbers* backward with negative subscripts. In fact if we try to extend the *associated left k- Fibonacci sequence* backward still keeping to the same rule, we get the following:

| $n$ | $A^L_{k,n}$ |
|---|---|
| -1 | $1-k$ |
| -2 | $1-k+k^2$ |
| -3 | $1-2k+k^2-k^3$ |
| -4 | $1-2k+3k^2-k^3+k^4$ |
| -5 | $1-3k+3k^2-4k^3+k^4-k^5$ |

Thus the sequence of *associated left k- Fibonacci numbers* is bilateral sequence, since it can be extended infinitely in both directions.

We next prove the divisibility property for $A^L_{k,n}$.

**Lemma 4.3.7** $A^L_{k,m} \mid A^L_{k,mn}$ for all non-zero integers $m, n$.

*Proof:* Let $m$ be fixed and we will proceed by inducting on $n$.

For $n = 1$. Then it is clear that $A^L_{k,m} \mid A^L_{k,m}$.

$\therefore$ The result is true for $n = 1$. Assume that the result is true for all $n = 1, 2, 3, \cdots, t$.

Thus $A^L_{k,m} \mid A^L_{k,mt}$ holds by assumption.

To prove the result is true for $n = t+1$. Using lemma 4.3.6, we get

$$A^L_{k,m(t+1)} = A^L_{k,mt+m}$$

$$= A^L_{k,mt-1}A^L_{k,m} + A^L_{k,mt}A^L_{k,m+1} - A^L_{k,mt+m-1}$$

$$= A^L_{k,mt-1}A^L_{k,m} + A^L_{k,mt}A^L_{k,m+1} - (A^L_{k,mt-1}A^L_{k,m-1} + A^L_{k,mt}A^L_{k,m} - A^L_{k,mt+m-2})$$

Continuously expand this expression; as by assumption $A^L_{k,m} \mid \mathrm{A}^L_{k,mt}$

$A^L_{k,m}$ divides the entire right side of the equation.

Hence $A^L_{k,m} \mid \mathrm{A}^L_{k,m(\mathrm{t}+1)}$. Thus result is true for all $n \geq 1$.

We next derive the formula for the sum of the squares of first *n associated left k-Fibonacci numbers*.

**Lemma 4.3.8** ${A^L_{k,1}}^2 + {A^L_{k,2}}^2 + {A^L_{k,3}}^2 + \cdots + {A^L_{k,\mathrm{n}}}^2 = \frac{1}{k}(A^L_{k,n}A^L_{k,n+1} - 1)$.

*Proof:* Since we have $A^L_{k,m} = \frac{1}{k}\left(A^L_{k,m+1} - A^L_{k,m-1}\right)$

We observe that ${A^L_{k,\mathrm{m}}}^2 = A^L_{k,\mathrm{m}}A^L_{k,\mathrm{m}}$

$$= A^L_{k,\mathrm{m}}\left[\frac{1}{k}\left(A^L_{k,m+1} - A^L_{k,m-1}\right)\right] = \frac{1}{k}\left(A^L_{k,m}A^L_{k,m+1} - A^L_{k,m}A^L_{k,m-1}\right)$$

Replacing $m = 1, 2, 3 \cdots n$, we have

$${A^L_{k,1}}^2 = \frac{1}{k}(\mathrm{A}^L_{k,1}\, A^L_{k,2} - \mathrm{A}^L_{k,1}\, A^L_{k,0})$$

$${A^L_{k,2}}^2 = \frac{1}{k}(\mathrm{A}^L_{k,2}\, A^L_{k,3} - A^L_{k,1}A^L_{k,2})$$

$${A^L_{k,3}}^2 = \frac{1}{k}(\mathrm{A}^L_{k,3}\, A^L_{k,4} - A^L_{k,2}A^L_{k,3})$$

$$\vdots$$

$${A^L_{k,\mathrm{n}-1}}^2 = \frac{1}{k}(\mathrm{A}^L_{k,\mathrm{n}-1}\, A^L_{k,\mathrm{n}} - A^L_{k,\mathrm{n}-2}A^L_{k,\mathrm{n}-1})$$

${A^L_{k,\mathrm{n}}}^2 = \frac{1}{k}(\mathrm{A}^L_{k,\mathrm{n}}\, A^L_{k,\mathrm{n}+1} - A^L_{k,\mathrm{n}-1}A^L_{k,\mathrm{n}})$. Adding all these equations, we get

$${A^L_{k,1}}^2 + {A^L_{k,2}}^2 + {A^L_{k,3}}^2 + \cdots + {A^L_{k,\mathrm{n}}}^2 = \frac{1}{k}(A^L_{k,n}A^L_{k,n+1} - \mathrm{A}^L_{k,1}\, A^L_{k,0}) = \frac{1}{k}(A^L_{k,n}A^L_{k,n+1} - 1).$$

The following result follows immediately from this lemma.

**Lemma 4.3.9** $A^L_{k,n}A^L_{k,n+1} \equiv 1(mod\, k)$.

*Proof:* We use Mathematical Induction to prove the result.

For $n = 1$, we have $A^L_{k,1}A^L_{k,2} = 1(1+k) = 1+k \equiv 1(mod\, k)$

Suppose it is true for $n = r$, Thus $F^R_{k,\mathrm{r}}F^R_{k,\mathrm{r+1}} \equiv 1(mod\, k)$ holds.

Now, $A^L_{k,\mathrm{r+1}}A^L_{k,\mathrm{r+2}} = A^L_{k,\mathrm{r+1}}\left(kA^L_{k,\mathrm{r+1}} + A^L_{k,\mathrm{r}}\right)$.

$$= kA^L_{k,\mathrm{r+1}}{}^2 + A^L_{k,\mathrm{r+1}}A^L_{k,\mathrm{r}} \equiv 1(mod\, k)$$

So the result is true for $n = r+1$ also. This proves the result for all integers *n*.

Similarly, we can prove the second result.

We finally prove the extended Cassini's identity.

**Lemma 4.3.10** $A^L_{k,\mathrm{n+1}}A^L_{k,\mathrm{n-1}} - A^L_{k,\mathrm{n}}{}^2 = k(-1)^{n+1}$.

*Proof:* We have $A^L_{k,\mathrm{n+1}}A^L_{k,\mathrm{n-1}} - A^L_{k,\mathrm{n}}{}^2 = (kA^L_{k,\mathrm{n}} + A^L_{k,\mathrm{n-1}})A^L_{k,\mathrm{n-1}} - A^L_{k,\mathrm{n}}{}^2$

$$= kA^L_{k,n}A^L_{k,n-1} - A^L_{k,n}{}^2 + A^L_{k,n-1}{}^2$$

$$= A^L_{k,\mathrm{n}}(kA^L_{k,\mathrm{n-1}} - A^L_{k,\mathrm{n}}) + A^L_{k,\mathrm{n-1}}{}^2$$

$$= -A^L_{k,\mathrm{n}}A^L_{k,\mathrm{n-2}} + A^L_{k,\mathrm{n-1}}{}^2$$

$$= (-1)(A^L_{k,\mathrm{n}}A^L_{k,\mathrm{n-2}} - A^L_{k,\mathrm{n-1}}{}^2).$$

Repeating the same process successively for right side, we get

$$\therefore A^L_{k,\mathrm{n+1}}A^L_{k,\mathrm{n-1}} - A^L_{k,\mathrm{n}}{}^2 = (-1)^1(A^L_{k,\mathrm{n}}A^L_{k,\mathrm{n-2}} - A^L_{k,\mathrm{n-1}}{}^2)$$

$$= (-1)^2(A^L_{k,\mathrm{n-1}}A^L_{k,\mathrm{n-3}} - A^L_{k,\mathrm{n-2}}{}^2)$$

$$= (-1)^3(A^L_{k,\mathrm{n-2}}A^L_{k,\mathrm{n-4}} - A^L_{k,\mathrm{n-3}}{}^2)$$

$$\vdots$$

$$= (-1)^n(A^L_{k,1}A^L_{k,-1} - A^L_{k,0}{}^2)$$

$$= (-1)^n(1(1-k) - 1)$$

$$= k(-1)^{n+1}.$$

## 4.4 Generating function of associated left *k*- Fibonacci numbers:

The *associated left k-Fibonacci number* which is defined as $A^L_{k,\text{n}+1} = kA^L_{k,\text{n}} + A^L_{k,\text{n}-1}, \quad n \geq 1$ with initial condition $A^L_{k,0} = 1$ is a second order difference equation with constant coefficient. Therefore, it has the characteristic equation $x^2 - kx - 1 = 0$.

**Lemma 4.4.1** The generating function for the generalized *associated left k- Fibonacci sequence* $\left\{A^L_{k,n}\right\}_{n=0}^{\infty}$ is given by $f(x) = \dfrac{1+x-kx}{1-kx-x^2}$ .

*Proof*: We begin with the formal power series representation of generating function for $\left\{A^L_{k,n}\right\}$ that is for $\left\{g_n\right\}$ .

$$f(x) = \sum_{m=0}^{\infty} A^L_{k,m} x^m = \sum_{m=0}^{\infty} g_m x^m = g_0 + g_1 x + g_2 x^2 + \cdots$$

$$= 1 + (1)x + (kg_1 + g_0)x^2 + (kg_2 + g_1)x^3 + (kg_3 + g_2)x^4 + \cdots$$

$$= 1 + x + kx(g_1 x + g_2 x^2 + \cdots) + x^2(g_0 + g_1 x + g_2 x^2 + \cdots)$$

$$= 1 + x + kx(\text{g}_0 + g_1 x + g_2 x^2 + \cdots) - kx + x^2(g_0 + g_1 x + g_2 x^2 + \cdots)$$

$$= 1 + x + kxf(x) + x^2 f(x) - kx$$

$$\therefore \; (1 - kx - x^2) f(x) = 1 + x - kx \Rightarrow f(x) = \frac{1+x-kx}{1-kx-x^2}.$$

This is the generating function for the generalized *associated left k- Fibonacci sequence* $\left\{A^L_{k,n}\right\}_{n=0}^{\infty}$.

## 4.5 The associated right *k*-Fibonacci numbers:

**Definition:** We define the sequence $\left\{A^R_{k,n}\right\}$ associate to right k-Fibonacci sequence $\left\{F^R_{k,n}\right\}$ as

$A^R_{k,0} = \dfrac{1}{k}$ and $A^R_{k,n} = F^R_{k,n} + F^R_{k,n-1}, \quad n = 1,2,3\ldots.$

We observe that the expression $A^R_{k,n}$ is the sum of the two consecutive *right k-Fibonacci numbers* $F^R_{k,n}$ and its predecessor $F^R_{k,n-1}$. The members of the sequence $\left\{A^R_{k,n}\right\}$ will be called *associated right k-Fibonacci numbers*. An equivalent definition for the sequence $\left\{A^R_{k,n}\right\}$ is

$$A^R_{k,n} = \begin{cases} \frac{1}{k} & ,if \ n=0 \\ 1 & ,if \ n=1 \\ 2F^R_{k,n-1} + kF^R_{k,n-2} & ,if \ n \geq 2 \end{cases}$$

Observe that $A^R_{k,n} = F^R_{k,n} + F^R_{k,n-1} = F^R_{k,n-1} + kF^R_{k,n-2} + F^R_{k,n-2} + kF^R_{k,n-3}$

$$= F^R_{k,n-1} + F^R_{k,n-2} + k(F^R_{k,n-2} + F^R_{k,n-3})$$

$$= A^R_{k,n-1} + kA^R_{k,n-2}$$

This allows defining recursively the sequence of *associated right k-Fibonacci numbers* as follows $A^R_{k,n} = \begin{cases} \frac{1}{k} & ,if \ n=0 \\ 1 & ,if \ n=1 \\ A^R_{k,n-1} + kA^R_{k,n-2} & ,if \ n \geq 2 \end{cases}$

Members of associated right k- Fibonacci sequence $\left\{A^R_{k,n}\right\}$ will be called *associated right k- Fibonacci numbers*. Some of them are

| $n$ | $A^R_{k,n}$ |
|---|---|
| 0 | $\frac{1}{k}$ |
| 1 | 1 |
| 2 | 2 |
| 3 | $2+k$ |
| 4 | $2+3k$ |
| 5 | $2+5k+k^2$ |

| | |
|---|---|
| 6 | $2+7k+4k^2$ |
| 7 | $2+9k+9k^2+k^3$ |
| 8 | $2+11k+16k^2+5k^3$ |
| 9 | $2+13k+25k^2+14k^3+k^4$ |
| 10 | $2+15k+36k^2+30k^3+6k^4$ |

Appendix-IV is computer program to obtain terms of *associated right k- Fibonacci sequence* $\{B_n\}$ using the programming language MATLAB (R2008a).

## 4.6 Basic identities of associated right k- Fibonacci numbers:

**Lemma 4.6.1** Sum of the first $n$ *associated right* $k$ *- Fibonacci numbers* is given by

$$A_{k,1}^R + A_{k,2}^R + A_{k,3}^R + \cdots + A_{k,n}^R = \frac{1}{k}(\mathrm{A}_{k,n+2}^R - 2).$$

*Proof:* From the recurrence relation of *associated right k- Fibonacci numbers*

$$A_{k,n}^R = A_{k,n-1}^R + kA_{k,n-2}^R, \quad n \geq 2 \qquad A_{k,0}^R = \frac{1}{k}, \; A_{k,1}^R = 1$$

$$A_{k,2}^R = A_{k,1}^R + kA_{k,0}^R$$
$$A_{k,3}^R = A_{k,2}^R + kA_{k,1}^R$$
$$\vdots$$
$$A_{k,n-2}^R = A_{k,n-3}^R + kA_{k,n-4}^R$$
$$A_{k,n-1}^R = A_{k,n-2}^R + kA_{k,n-3}^R$$
$$A_{k,n}^R = A_{k,n-1}^R + kA_{k,n-2}^R$$

Adding all these equations term by term, we get

$$A_{k,1}^R + A_{k,2}^R + A_{k,3}^R + \cdots + A_{k,n}^R$$
$$= A_{k,1}^R + (1+\mathrm{k})\left(\mathrm{A}_{k,1}^R + A_{k,2}^R + \cdots + A_{k,\mathrm{n}-2}^R\right) + \mathrm{kA}_{k,0}^R + A_{k,\mathrm{n}-1}^R$$
$$= A_{k,1}^R + kA_{k,0}^R + (1+\mathrm{k})\left(\mathrm{A}_{k,1}^R + A_{k,2}^R + \cdots + A_{k,n}^R\right) - kA_{k,n-1}^R - (1+\mathrm{k})\,\mathrm{A}_{k,n}^R$$
$$\therefore \quad -k(\mathrm{A}_{k,1}^R + A_{k,2}^R + \cdots + A_{k,n}^R) = A_{k,2}^R - (kA_{k,n-1}^R + A_{k,n}^R + kA_{k,\mathrm{n}}^R)$$

$$= 2 - (A^R_{k,n+1} + kA^R_{k,n}) = 2 - A^R_{k,n+2}$$

$$\therefore \quad A^R_{k,1} + A^R_{k,2} + A^R_{k,3} + \cdots + A^R_{k,n} = \frac{1}{k}(A^R_{k,n+2} - 2).$$

**Lemma 4.6.2** $A^R_{k,1} + A^R_{k,2} + A^R_{k,3} + \cdots + A^R_{k,2n} = \frac{1}{k}(A^R_{k,2n+2} - 2)$.

*Proof:* $A^R_{k,n} = A^R_{k,n-1} + kA^R_{k,n-2}, \quad n \geq 2 \qquad A^R_{k,0} = \frac{1}{k}, \; A^R_{k,1} = 1$

$$A^R_{k,2} = A^R_{k,1} + kA^R_{k,0}$$

$$A^R_{k,3} = A^R_{k,2} + kA^R_{k,1}$$

$$\vdots$$

$$A^R_{k,2n-1} = A^R_{k,2n-2} + kA^R_{k,2n-3}$$

$$A^R_{k,2n} = A^R_{k,2n-1} + kA^R_{k,2n-2}$$

Adding all these equations term by term, we get

$$A^R_{k,1} + A^R_{k,2} + A^R_{k,3} + \cdots + A^R_{k,2n}$$

$$= A^R_{k,1} + (1+k)(A^R_{k,1} + A^R_{k,2} + \cdots + A^R_{k,2n-2}) + kA^R_{k,0} + A^R_{k,2n-1}$$

$$= A^R_{k,1} + k\left(\frac{1}{k}\right) + (1+k)(A^R_{k,1} + A^R_{k,2} + \cdots + A^R_{k,2n})$$

$$- (1+k)(A^R_{k,2n-1} + A^R_{k,2n}) + A^R_{k,2n-1}$$

$$\therefore \; -k(A^R_{k,1} + A^R_{k,2} + \cdots + A^R_{k,2n}) = 2 - (1+k)(A^R_{k,2n-1} + A^R_{k,2n}) + A^R_{k,2n-1}$$

$$\therefore \quad -k(A^R_{k,1} + A^R_{k,2} + \cdots + A^R_{k,2n}) = 2 - (A^R_{k,2n} + kA^R_{k,2n-1}) - kA^R_{k,2n}$$

$$\therefore \quad -k(A^R_{k,1} + A^R_{k,2} + \cdots + A^R_{k,2n}) = 2 - (A^R_{k,2n+1} + kA^R_{k,2n})$$

$$\therefore \quad A^R_{k,1} + A^R_{k,2} + A^R_{k,3} + \cdots + A^R_{k,n} = \frac{1}{k}(A^R_{k,2n+2} - 2).$$

This proves the lemma.

The following results follow immediately from above two lemmas.

**Lemma 4.6.3** $A^R_{k,n+2} \equiv 2 (\bmod m)$ and $A^R_{k,2n+2} \equiv 2 (\bmod m)$ .

We next obtain the formulas for sum of first *n associated right k-Fibonacci numbers* with only odd or even subscripts.

**Lemma 4.6.4** $A^R_{k,1}+A^R_{k,3}+A^R_{k,5}+\cdots+A^R_{k,2n-1}=\dfrac{1}{k(2-k)}(\mathrm{A}^R_{k,2n+2}-kA^R_{k,2n+1}+k-2)$

*Proof:* We have $A^R_{k,n}=A^R_{k,n-1}+kA^R_{k,n-2},\quad n\geq 2 \qquad A^R_{k,0}=\dfrac{1}{k},\ A^R_{k,1}=1$

$$A^R_{k,3}=A^R_{k,2}+kA^R_{k,1}$$

$$A^R_{k,5}=A^R_{k,4}+kA^R_{k,3}$$

$$A^R_{k,7}=A^R_{k,6}+kA^R_{k,5}$$

$$\vdots$$

$$A^R_{k,2n-1}=A^R_{k,2n-2}+kA^R_{k,2n-3}$$

Adding all these equations term by term,

$$A^R_{k,1}+A^R_{k,3}+A^R_{k,5}+\cdots+A^R_{k,2n-1}$$

$$=A^R_{k,1}+(\mathrm{A}^R_{k,2}+A^R_{k,4}+\cdots+A^R_{k,2n-2})+k(\mathrm{A}^R_{k,1}+A^R_{k,3}+\cdots+A^R_{k,2n-3})$$

$$\therefore\ 2(\mathrm{A}^R_{k,1}+A^R_{k,3}+A^R_{k,5}+\cdots+A^R_{k,2n-1})=1+(\mathrm{A}^R_{k,1}+A^R_{k,2}+\cdots+A^R_{k,2n-1}+A^R_{k,2n})$$

$$+k(\mathrm{A}^R_{k,1}+A^R_{k,3}+A^R_{k,5}+\cdots+A^R_{k,2n-1})-A^R_{k,2n}-kA^R_{k,2n-1}$$

$$\therefore (2-k)(\mathrm{A}^R_{k,1}+A^R_{k,3}+A^R_{k,5}+\cdots+A^R_{k,2n-1})=1+\frac{1}{k}(\mathrm{A}^R_{k,2n+2}-2)-A^R_{k,2n}-kA^R_{k,2n-1}$$

$$=\frac{1}{k}(\mathrm{A}^R_{k,2n+2}+k-2)-A^R_{k,2n+1}$$

$$\therefore\ A^R_{k,1}+A^R_{k,3}+A^R_{k,5}+\cdots+A^R_{k,2n-1}=\frac{1}{k(2-k)}(\mathrm{A}^R_{k,2n+2}-kA^R_{k,2n+1}+k-2)$$

**Lemma 4.6.5** $A^R_{k,2}+A^R_{k,4}+A^R_{k,6}+\cdots+A^R_{k,2n}=\dfrac{1}{k(2-k)}(\mathrm{A}^R_{k,2n+2}-k^2A^R_{k,2n}+k-2)$

*Proof:* We have $A^R_{k,n}=A^R_{k,n-1}+kA^R_{k,n-2},\quad n\geq 2 \qquad A^R_{k,0}=\dfrac{1}{k},\ A^R_{k,1}=1$

$$A^R_{k,2}=A^R_{k,1}+kA^R_{k,0}$$

$$A^R_{k,4}=A^R_{k,3}+kA^R_{k,2}$$

$$A^R_{k,6}=A^R_{k,5}+kA^R_{k,4}$$

$$\vdots$$

$$A^R_{k,2n}=A^R_{k,2n-1}+kA^R_{k,2n-2}$$

Adding all these equations term by term,

$A_{k,2}^R + A_{k,4}^R + A_{k,6}^R + \cdots + A_{k,2n}^R$

$$= (\mathrm{A}_{k,1}^R + A_{k,3}^R + \cdots + A_{k,2n-1}^R) + \mathrm{k}\, A_{k,0}^R + k(\mathrm{A}_{k,2}^R + A_{k,4}^R + \cdots + A_{k,2n-2}^R)$$

$$\therefore \quad 2(\mathrm{A}_{k,2}^R + A_{k,4}^R + \cdots + A_{k,2n}^R) = (\mathrm{A}_{k,1}^R + A_{k,2}^R + \cdots + A_{k,2n}^R) + \mathrm{k}(\frac{1}{k})$$
$$+ k(\mathrm{A}_{k,2}^R + A_{k,4}^R + \cdots + A_{k,2n}^R) - kA_{k,2n}^R$$

$$\therefore \quad (2-k)(\mathrm{A}_{k,2}^R + A_{k,4}^R + \cdots + A_{k,2n}^R) = \frac{1}{k}(\mathrm{A}_{k,2\mathrm{n}+2}^R - 2) + 1 - kA_{k,2n}^R$$

$$= \frac{1}{k}(\mathrm{A}_{k,2n+2}^R - k^2 A_{k,2n}^R + k - 2)$$

$$\therefore \quad A_{k,2}^R + A_{k,4}^R + \cdots + A_{k,2n}^R = \frac{1}{k(2-k)}(\mathrm{A}_{k,2n+2}^R - k^2 A_{k,2n}^R + k - 2).$$

Now we obtain the value of multiplication of two consecutive *associated right* $k$ *-Fibonacci numbers*.

**Lemma 4.6.6** $A_{k,n}^R\, A_{k,n+1}^R = A_{k,n}^{R\ 2} + kA_{k,n-1}^{R\ \ 2} + k^2 A_{k,n-2}^{R\ \ 2} + \cdots + k^{n-1} A_{k,1}^{R\ 2} + k^n A_{k,0}^R A_{k,1}^R$

$$= 2k^{n-1} + \sum_{r=2}^{n} k^{n-r} A_{k,\mathrm{r}}^{R\ 2}. \qquad .$$

*Proof:* We have $A_{k,n}^R = A_{k,n-1}^R + kA_{k,n-2}^R$ and $A_{k,n+1}^R = A_{k,n}^R + kA_{k,n-1}^R$

Now $A_{k,n}^R A_{k,n+1}^R = A_{k,n}^R(\mathrm{A}_{k,n}^R + kA_{k,n-1}^R) = A_{k,n}^{R\ 2} + kA_{k,n-1}^R(\mathrm{A}_{k,n-1}^R + kA_{k,n-2}^R)$

$$= A_{k,n}^{R\ 2} + kA_{k,n-1}^{R\ \ 2} + \mathrm{k}^2\, A_{k,n-2}^R(\mathrm{A}_{k,n-2}^R + kA_{k,n-3}^R)$$

$$= A_{k,n}^{R\ 2} + kA_{k,n-1}^{R\ \ 2} + \mathrm{k}^2\, A_{k,n-2}^{R\ \ 2} + \mathrm{k}^3\, A_{k,n-3}^R(\mathrm{A}_{k,n-3}^R + kA_{k,n-4}^R)$$
$$= A_{k,n}^{R\ 2} + kA_{k,n-1}^{R\ \ 2} + \mathrm{k}^2\, A_{k,n-2}^{R\ \ 2} + \cdots + k^{n-1} A_{k,1}^R(\mathrm{A}_{k,1}^R + A_{k,0}^R)$$

$$= A_{k,n}^{R\ 2} + kA_{k,n-1}^{R\ \ 2} + \mathrm{k}^2\, A_{k,n-2}^{R\ \ 2} + \cdots + k^{n-2} F_{k,2}^{R\ 2} + k^{n-1} A_{k,1}^{R\ 2} + k^n A_{k,1}^R A_{k,0}^R$$

$$= A_{k,n}^{R\ 2} + kA_{k,n-1}^{R\ \ 2} + \mathrm{k}^2\, A_{k,n-2}^{R\ \ 2} + \cdots + k^{n-2} F_{k,2}^{R\ 2} + k^{n-1}(1)^2 + k^n(1)(1/k)$$

$$= 2k^{n-1} + \sum_{r=2}^{n} k^{n-r} A_{k,\mathrm{r}}^{R\ 2}.$$

We now derive reduction formula for $A_{k,n}^R$.

**Lemma 4.6.7** $A_{k,m+n}^R = kA_{k,m-1}^R A_{k,n}^R + A_{k,m}^R A_{k,n+1}^R - A_{k,m+n-1}^R$

*Proof:* Let $m$ be fixed and we will proceed by inducting on $n$.

Also we know value $A^R_{k,0} = \frac{1}{k}$, $A^R_{k,1} = 1$, $A^R_{k,2} = 2$

Take $n = 0$. Then $RHS = kA^R_{k,\text{m}-1}A^R_{k,0} + A^R_{k,\text{m}}A^R_{k,1} - A^R_{k,\text{m}-1} = A^R_{k,\text{m}}$

When $n = 1$, we have

$$A^R_{k,\text{m}+1} = kA^R_{k,\text{m}-1}A^R_{k,1} + A^R_{k,\text{m}}A^R_{k,2} - A^R_{k,\text{m}} \quad \therefore \quad A^R_{k,\text{m}+1} = A^R_{k,\text{m}} + kA^R_{k,\text{m}-1}$$

This is obviously true.

Now let us assume that the result is true for positive integers $n = 1, 2, 3, \cdots, t$

$$A^R_{k,\text{m}+t} = kA^R_{k,\text{m}-1}A^R_{k,t} + A^R_{k,\text{m}}A^R_{k,t+1} - A^R_{k,\text{m+t}-1} \text{ and}$$

$$A^R_{k,\text{m}+(t-1)} = kA^R_{k,\text{m}-1}A^R_{k,t-1} + A^R_{k,\text{m}}A^R_{k,t} - A^R_{k,\text{m+t}-2}$$

We will show that it holds for $n = t + 1$. Now from above two equations , we have

$$\begin{aligned} A^R_{k,\text{m}+t} + kA^R_{k,m+(t-1)} &= kA^R_{k,\text{m}-1}(\text{A}^R_{k,t} + kA^R_{k,t-1}) + A^R_{k,m}(\text{A}^R_{k,t+1} + kA^R_{k,t}) \\ &\quad - (A^R_{k,\text{m+t}-1} + kA^R_{k,\text{m+t}-2}) \\ &= kA^R_{k,\text{m}-1}A^R_{k,t+1} + A^R_{k,\text{m}}A^R_{k,t+2} - A^R_{k,\text{m+t}} \\ &= kA^R_{k,\text{m}-1}A^R_{k,t+1} + A^R_{k,\text{m}}A^R_{k,(t+1)+1} - A^R_{k,\text{m+(t+1)}-1} = A^R_{k,\text{m}+(t+1)}. \end{aligned}$$

This is true for $n = t + 1$. This proves the lemma.

It is often useful to extend the sequence of *associated right k- Fibonacci numbers* backward with negative subscripts. In fact if we try to extend the *associated right k- Fibonacci sequence* backward still keeping to the same rule, we get the following:

| $n$ | $A^R_{k,\text{n}}$ |
|---|---|
| -1 | $-\frac{1-k}{k^2}$ |
| -2 | $\frac{1}{k^3}$ |
| -3 | $-\frac{1+k-k^2}{k^4}$ |
| -4 | $\frac{1+2k-k^2}{k^5}$ |
| -5 | $-\frac{1+3k-k^3}{k^6}$ |

Such *associated right k- Fibonacci sequence* can be extended infinitely in both directions is called Bilateral.

We next prove the divisibility property for $A^R_{k,\mathrm{n}}$.

**Lemma 4.6.8** $A^R_{k,m} \mid \mathrm{A}^R_{k,mn}$ for all integers $m, n$.

*Proof:* Let $m$ be fixed and we will proceed by inducting on $n$.

If either $m$ or $n$ equal to zero, then the result is true.

Let for $n=1$, it is clear that $A^R_{k,m} \mid \mathrm{A}^R_{k,m}$.

$\therefore$ The result is true for $n=1$.

Assume that the result is true for $n = 1, 2, 3, \cdots t$

Thus $A^R_{k,m} \mid \mathrm{A}^R_{k,mt}$ holds by assumption.

To prove the result is true for $n = t+1$. Now using lemma 4.6.7, we get

$$A^R_{k,m(t+1)} = A^R_{k,mt+m} = kA^R_{k,mt-1}A^R_{k,m} + A^R_{k,mt}A^R_{k,m+1} - A^R_{k,mt+m-1}$$
$$= kA^R_{k,mt-1}A^R_{k,m} + A^R_{k,mt}A^R_{k,m+1} - \left(kA^R_{k,mt-1}A^R_{k,m-1} + A^R_{k,mt}A^R_{k,m} - A^R_{k,mt+m-2}\right)$$

Continuing this process, Since by assumption $A^R_{k,m}$ divides the entire right side of the equation.

Hence $A^R_{k,m} \mid \mathrm{A}^R_{k,m\,(t+1)}$. This proves the result for $n \geq 1$.

We next establish the relation for sum of squares consecutive *associated right k-Fibonacci numbers*.

**Lemma 4.6.9** $A^{R\;2}_{k,n} + \frac{1}{k} A^{R}_{k,n+1}{}^2 = \frac{1}{k}(A^R_{k,2n+1} + A^R_{k,2n})$.

*Proof*: We prove this result by the principal of mathematical induction.

For $n=1$, we have

$$LHS = A^{R\;2}_{k,1} + \frac{1}{k} A^{R\;2}_{k,2} = 1 + \frac{1}{k}(4) = \frac{1}{k}(k+4) = \frac{1}{k}(A^R_{k,3} + A^R_{k,2}) = RHS.$$

This proves the result for $n=1$.

We assume that it is true for all integers up to some positive integer ‘$t$’.

$$\therefore \quad A^{R\;2}_{k,t} + \frac{1}{k} A^{R}_{k,t+1}{}^2 = \frac{1}{k}(A^R_{k,2t+1} + A^R_{k,2t}) \text{ holds by assumption.}$$

Now $A^R_{k,t+1}{}^2 + \frac{1}{k} A^R_{k,t+2}{}^2 = A^R_{k,t+1}{}^2 + \frac{1}{k}(A^R_{k,t+1} + kA^R_{k,t})^2$

$$= {A^R_{k,t+1}}^2 + \frac{1}{k}({A^R_{k,t+1}}^2 + 2kA^R_{k,t}A^R_{k,t+1} + k^2{A^R_{k,t}}^2)$$

$$= {A^R_{k,t+1}}^2 + k{A^R_{k,\mathrm{t}}}^2 + \frac{1}{k}({A^R_{k,t+1}}^2 + kA^R_{k,t}A^R_{k,t+1} + kA^R_{k,t}A^R_{k,t+1})$$

$$= k({A^R_{k,t}}^2 + \frac{1}{k}{A^R_{k,t+1}}^2) + \frac{1}{k}[A^R_{k,t+1}(A^R_{k,t+1} + kA^R_{k,t}) + kA^R_{k,t}A^R_{k,t+1}]$$

$$= k\frac{1}{k}(A^R_{k,2t+1} + A^R_{k,2t}) + \frac{1}{k}[A^L_{k,t+1}A^L_{k,t+2} + kA^L_{k,t}A^L_{k,t+1}]$$

$$= A^R_{k,2t+1} + A^R_{k,2t} + \frac{1}{k}(\mathrm{k}\,A^R_{k,t}A^R_{k,t+1} + A^R_{k,t+1}A^R_{k,t+2})$$

Since from lemma 4.6.7

$$\therefore\quad {A^R_{k,t+1}}^2 + \frac{1}{k}{A^R_{k,t+2}}^2 = A^R_{k,2t+1} + A^R_{k,2t} + \frac{1}{k}(A^R_{k,t+1+t+1} + A^R_{k,t+1+t+1-1})$$

$$= \frac{1}{k}(A^R_{k,2t+2} + kA^R_{k,2t+1} + A^R_{k,2t+1} + kA^R_{k,2t})$$

$$= \frac{1}{k}(A^R_{k,2t+3} + A^R_{k,2t+2}) = \frac{1}{k}(A^R_{k,2(t+1)+1} + A^R_{k,2(t+1)})$$

This proves the result by induction.

We finally prove the analogous of one of the oldest identities involving the Fibonacci numbers – Cassini's identity, which was discovered in 1680 by a French astronomer Jean-Dominique Cassini.

**Lemma 4.6.10** $A^R_{k,n+1}.\mathrm{A}^R_{k,n-1} - {A^R_{k,n}}^2 = (-k)^{n-2}(k-2)$.

*Proof:* $A^R_{k,n+1}.\mathrm{A}^R_{k,n-1} - {A^R_{k,n}}^2 = (\mathrm{A}^R_{k,n} + kA^R_{k,n-1})\,\mathrm{A}^R_{k,n-1} - {A^R_{k,n}}^2$

$$= A^R_{k,n}A^R_{k,n-1} - {A^R_{k,n}}^2 + k{A^R_{k,n-1}}^2$$

$$= A^R_{k,n}(\mathrm{A}^R_{k,n-1} - A^R_{k,n}) + k{A^R_{k,n-1}}^2$$

$$= A^R_{k,n}(-kA^R_{k,n-2}) + k{A^R_{k,n-1}}^2$$

$$= -k(\mathrm{A}^R_{k,n}\,A^R_{k,n-2} - {A^R_{k,n-1}}^2)$$

We can repeat the above process on the right side.

$$\therefore\quad A^R_{k,n+1}.\mathrm{A}^R_{k,n-1} - {A^R_{k,n}}^2 = (-k)^1(\mathrm{A}^R_{k,n}.\mathrm{A}^R_{k,n-2} - {A^R_{k,n-1}}^2)$$

$$= (-k)^2(\mathrm{A}^R_{k,n-1}.\mathrm{A}^R_{k,n-3} - {A^R_{k,n-2}}^2)$$

$$\vdots$$

$$= (-k)^n (A^R_{k,1} . A^R_{k,-1} - A^{R\ 2}_{k,0})$$

$$= (-k)^n (1 . \frac{k-1}{k^2} - \frac{1}{k^2}) = (-k)^{n-2} (k-2)$$

Since we have $A^R_{k,1} = 1, A^R_{k,0} = \frac{1}{k}, A^R_{k,-1} = \frac{k-1}{k^2}$

$\therefore \quad A^R_{k,n+1} . A^R_{k,n-1} - A^{R\ 2}_{k,n} = (-k)^{n-2} (k-2)$.

**Lemma 4.6.11** $A^R_{k,n} = 1 + k \sum_{i=0}^{n-2} A^R_{k,i}; \; n \geq 2$.

*Proof:* We prove this result by the principal of mathematical induction.

For $n = 2$, we have $LHS = A^R_{k,2} = 2 = 1 + k\left(\frac{1}{k}\right) = 1 + kA^R_{k,0} = RHS$.

This proves the result for $n = 2$.

We assume that it is true for all integers up to some positive integer '$t$'.

Thus $A^R_{k,t} = 1 + k \sum_{i=0}^{t-2} A^R_{k,i}; \; t \geq 2$, holds by assumption.

Now we consider the right side of the result to be proved for $n = t + 1$.

$$RHS = 1 + k \sum_{i=0}^{t-1} A^R_{k,i} = 1 + k\left(A^R_{k,0} + A^R_{k,1} + A^R_{k,2} \cdots + A^R_{k,t-1}\right)$$

$$= A^R_{k,1} + kA^R_{k,0} + k(A^R_{k,1} + A^R_{k,2} \cdots + A^R_{k,t-1})$$

$$= A^R_{k,2} + kA^R_{k,1} + k(A^R_{k,2} + A^R_{k,3} + \cdots + A^R_{k,t-1})$$

$$= A^R_{k,3} + kA^R_{k,2} + k(A^R_{k,3} + A^R_{k,4} + \cdots + A^R_{k,t-1}).$$

Continuing this process on right hand side, we get

$$\therefore \; RHS = F^R_{k,t} + kF^R_{k,t-1} = F^R_{k,t+1} = LHS$$

$\therefore$ The result is true for $n = t + 1$. This proves the result by induction.

**Lemma 4.6.12** $\gcd\left(A^R_{k,n}, A^R_{k,n+1}\right) = 1, \; \forall \; n = 0, 1, 2, 3, \ldots$

*Proof*: Suppose that $A^R_{k,n}$ and $A^R_{k,n+1}$ are both divisible by a positive integer *d*.

Then clearly $A^R_{k,n+1} - A^R_{k,n} = A^R_{k,n} + kA^R_{k,n-1} - A^R_{k,n} = kA^R_{k,n-1}$

will also be divisible by *d*. Then right hand side of this result is divisible by *d*.

This gives $d / kA^R_{k,n-1}$.

First we claim that $A^R_{k,n}$ is always relatively prime to *k*.

From the lemma 4.6.11, we have $A^R_{k,n} = 1 + k\sum_{i=0}^{n-2} A^R_{k,i};\ n \geq 2$.

If some integer $d' > 1$ is divisor of both $A^R_{k,n}$ and *k*, then from above result it is clear that $d'$ must divide 1, a contradiction. Thus $A^R_{k,n}$ is always relatively prime to $k \Rightarrow d \mid \mathrm{A}^R_{k,n-1}$.

Continuing this argument we see that $d \mid \mathrm{A}^R_{k,n-2}$, $d \mid \mathrm{A}^R_{k,n-3}$ and so on. Eventually, we must have $d \mid \mathrm{A}^R_{k,1}$. Since $A^R_{k,1} = 1$ we get $d = 1$, this proves the required result.

## 4.7 Generating function of associated right k- Fibonacci numbers:

**Lemma 4.7.1** The generating function for the generalized *associated right k- Fibonacci sequence* $\left\{A^R_{k,n}\right\}_{n=0}^{\infty}$ is given by $f(x) = \dfrac{1+(k-1)x}{k(1-x-kx^2)}$.

*Proof:* We begin with the formal power series representation of generating function for $\left\{A^R_{k,n}\right\}$ that is for $\left\{g_n\right\}$.

$$f(x) = \sum_{m=0}^{\infty} A^R_{k,m} x^m = \sum_{m=0}^{\infty} g_m x^m = g_0 + g_1 x + g_2 x^2 + \cdots$$

$$= \frac{1}{k} + (1)x + (g_1 + kg_0)x^2 + (g_2 + kg_1)x^3 + (g_3 + kg_2)x^4 + \cdots$$

$$= \frac{1}{k} + x + x(g_1 x + g_2 x^2 + \cdots) + kx^2(g_0 + g_1 x + g_2 x^2 + \cdots)$$

$$= \frac{1}{k} + x + x(g_0 + g_1 x + g_2 x^2 + \cdots) - \frac{x}{k} + kx^2(g_0 + g_1 x + g_2 x^2 + \cdots)$$

$$= \frac{1+kx}{k} + xf(x) + kx^2 f(x) - \frac{x}{k}$$

$$\therefore\ (1 - x - kx^2) f(x) = \frac{1+kx}{k} - \frac{x}{k} \Rightarrow f(x) = \frac{1+(k-1)x}{k(1-x-kx^2)}.$$

This is the generating function for the generalized *associated right k- Fibonacci sequence* $\left\{A^R_{k,n}\right\}_{n=0}^{\infty}$.

# Chapter- 5

# Golden proportions for the Generalized left and right k-Fibonacci numbers

It is known fact that the ratios of consecutive terms of *Fibonacci sequence* converges to the fixed ratio **[12, 43]**. In this chapter we consider the further generalization of recursive formula of *k-Fibonacci numbers*. We derive the '*golden proportion*' for the whole family of this new generalized sequence.

## 5.1 Introduction:

It is well-known that the ratio of consecutive terms of a *Fibonacci sequence* converges to Golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ which is the positive root of the equation $x^2 - x - 1 = 0$.

Stakhov **[39, 41]** defined the $p-Fibonacci\ numbers\ F_p(\mathrm{n})$ by the recurrence relation

$$F_p(n) = \begin{cases} 1 & ;1 \le n \le p+1 \\ F_p(n-1) + F_p(n-p-1) & ;n > p+1 \end{cases};$$

where $n = 1, 2, 3, \cdots$

It can be seen that by taking $p = 1$, the recurrence relation becomes $F(n) = F(n-1) + F(n-2),\ n \ge 2;\ F(1) = F(2) = 1$, which is a well known *Fibonacci sequence*. The values of $F_p(n)$ for $p = 1, 2, ...., 5$ and for first 15 values of *n* are shown below for the ready reference.

| ***n*** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** | **13** | **14** | **15** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ***F₁(n)*** | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | 233 | 377 | 610 |
| ***F₂(n)*** | 1 | 1 | 1 | 2 | 3 | 4 | 6 | 9 | 13 | 19 | 28 | 41 | 60 | 88 | 129 |
| ***F₃(n)*** | 1 | 1 | 1 | 1 | 2 | 3 | 4 | 5 | 7 | 10 | 14 | 19 | 26 | 36 | 50 |
| ***F₄(n)*** | 1 | 1 | 1 | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 11 | 15 | 20 | 26 |
| ***F₅(n)*** | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 12 | 16 |

**[Table 1: Values of $F_p(n)$]**

Stakhov also shown that $F_p(n)$ satisfies $\lim_{n \to \infty} \frac{\mathrm{F_p(n)}}{\mathrm{F_p(n-1)}} = \phi_\mathrm{p}$, where the *golden p-proportion* $\phi_p$ is the root of $x^{p+1} = x^p + 1$.

De Villiers **[9]** generalized the recursive formula to $F_{n+t+1} = F_{n+t} + F_n$, where $t = 1, 2, 3, \ldots$; and made the interesting discovery that for each member of this family, the ratios of consecutive terms converge to the positive roots of $x^{t+1} - x^{t} - 1 = 0$ .

However, based on the assumption that $\lim_{x\to\infty}\frac{F_{n+t+1}}{F_{n+t}}$ exists, only a partial proof to this result was given. He suggested a simple proof for the case where *k* is even.

Later Falcon **[14]** considered the same problem and provided the complete proof of the problem. In this chapter we generalize the above recurrence relation and consider the problem of finding the 'golden proportion' for the particular family of this generalized sequence.

## 5.2 Some preliminaries of left *k*- Fibonacci numbers:

First we consider the particular generalization of recursive formula of $F_{k,n}^{L}$ as $G_{n+a+1} = kG_{n+a} + G_n$; where $a$ is any positive integer. Clearly $G_n = F_{k,n}^{L}$ when $G_0 = 0,\ G_1 = 1$.If we consider the *left k-Fibonacci numbers* for the case $k = 2,\ 3,\ \ldots$ then we observe that ratio of consecutive *left k-Fibonacci numbers* converge to a fixed ratio. This fact is presented in the following table for $k = 2, 3$:

| **n** | $G_{2,n}$ | $\frac{G_{2,n+1}}{G_{2,n}}$ | $G_{3,n}$ | $\frac{G_{3,n+1}}{G_{3,n}}$ |
|---|---|---|---|---|
| **0** | 0 | | 0 | |
| **1** | 1 | 2 | 1 | 3 |
| **2** | 2 | 2.5 | 3 | 3.33 |
| **3** | 5 | 2.4 | 10 | 3.3 |
| **4** | 12 | 2.416 | 33 | 3.3030 |
| **5** | 29 | 2.413793103 | 109 | 3.302752 |
| **6** | 70 | 2.414285714 | 360 | 3.30277 |
| **7** | 169 | 2.414201183 | 1189 | 3.302775 |
| **8** | 408 | 2.414215686 | 3927 | 3.302775 |
| **9** | 985 | 2.414213198 | 12970 | 3.302775 |
| **10** | 2378 | 2.414213625 | 42837 | 3.302775 |
| **11** | 5741 | 2.414213557 | 141481 | **3.302775** |

**[Table 2: Values of Ratio]**

We consider the more generalized recurrence relation given by $G_{n+a+1} = kG_{n+a} + G_n$; where $a$ is any positive integer. By substituting $n$ for $n+a+1$, this recurrence relation becomes

$$G_n = kG_{n-1} + G_{n-a-1} \tag{5.2.1}$$

This is a difference equation with characteristic equation $x^n = kx^{n-t} + x^{n-a-1}$.
This is same as

$$x^{a+1} = kx^a + 1\,. \tag{5.2.2}$$

We write it as $x^a(x-k) = 1$, which implies $x^a = \dfrac{1}{x-k}$.

Thus solving (5.2.2) is equivalent to solving the system

$$f(x) = \frac{1}{x-k};\;\; g(x) = x^a \tag{5.2.3}$$

This is same as finding intersection of these curves defined for $a = 1, 2, 3, \ldots$ We now consider the different cases when $a$ is even or odd.

If $a$ is even, the graph of $g(x) = x^a$ is a parabolic type curve. In this case, the intersection of two curves of (5.2.3) is shown in figures 1 and 2.

Also if $a$ is odd, the graph of $g(x) = x^a$ is a curve of cubic type. In this case, the intersection of two curves of (5.2.3) is shown in figures 3 and 4.

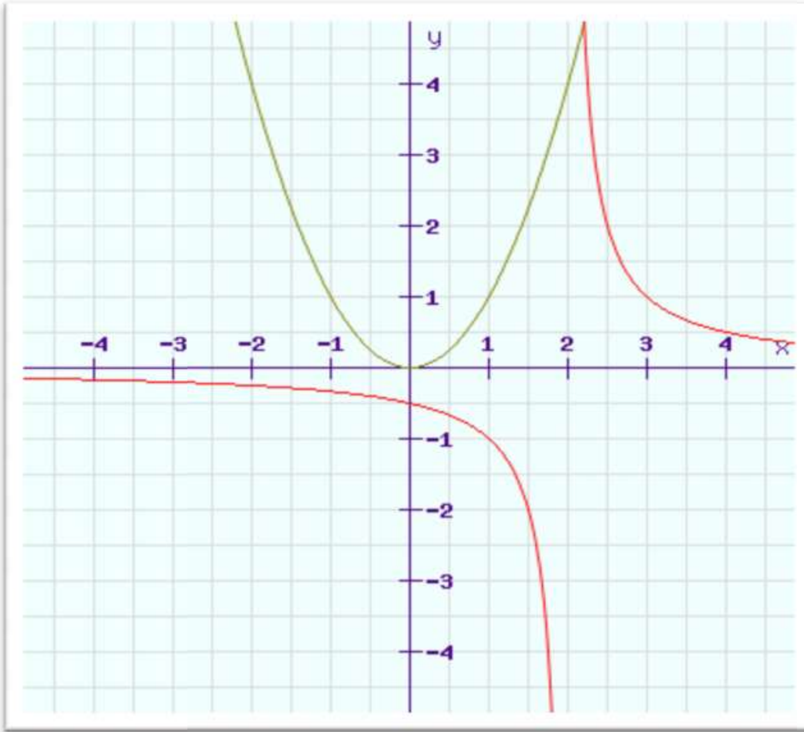



**Figure-1 [*a* even, *k* even]**

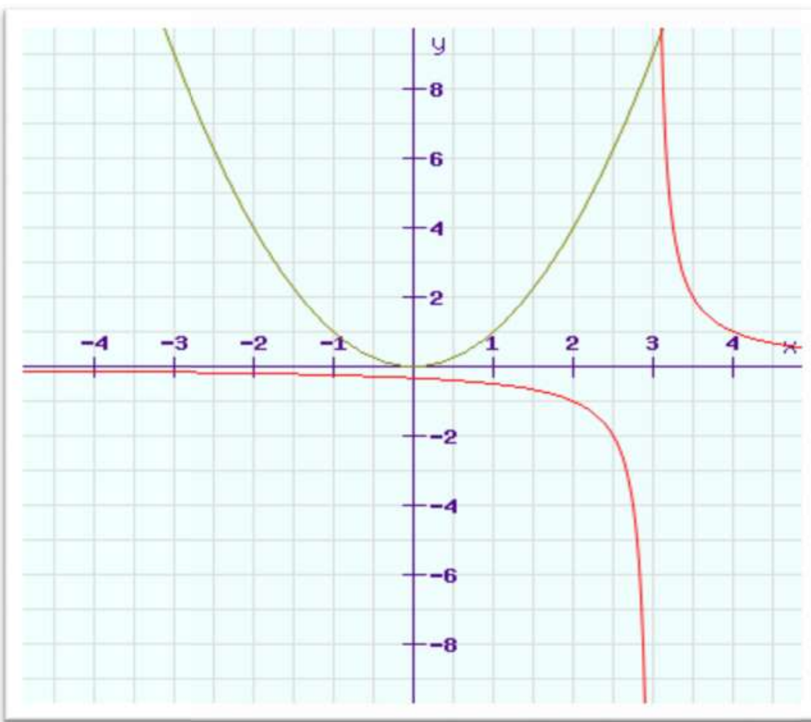



**Figure-2 [*a* even, *k* odd]**

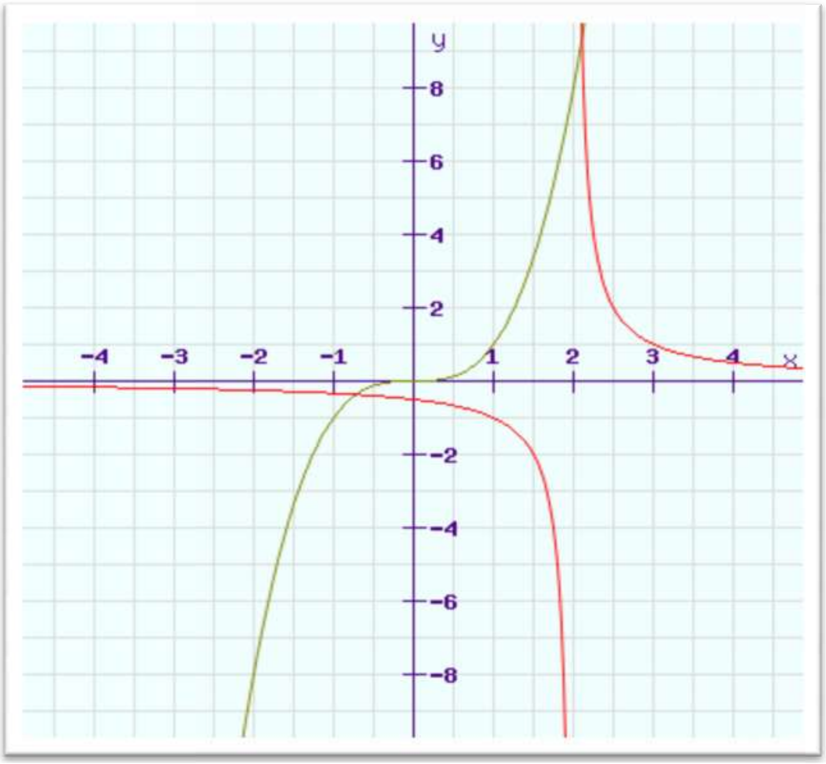



**Figure-3 [*a* odd, *k* even]**

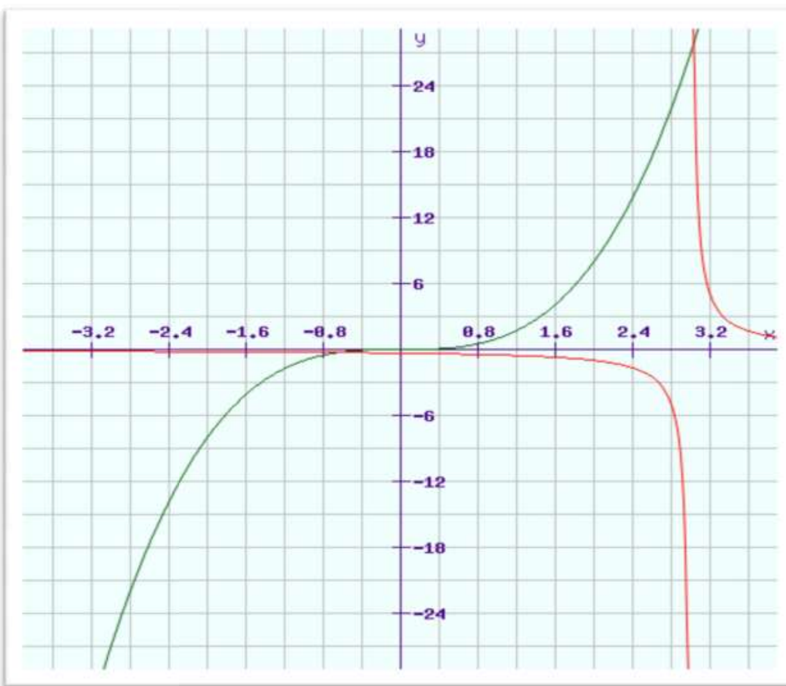



**Figure-4 [a is odd, *k* odd]**

It is clear from these figures that when $a$ is even, both $f(x)$ and $g(x)$ are symmetric about the y-axis. In this case, system (5.2.3) has one real roots $x = M$ ; for some real number M such that, $M > k$.

Also when $a$ is odd, system (5.2.3) has two real roots $x = M, -M_1$, where $0 < M_1 < 1$ and $M > k$.

We summarize this in the following table.

| ***a*** | ***k*** | **Number of Intersecting Points** | **Roots** |
|---|---|---|---|
| **Even** | **Even** | 1 | $x = M;\ M > k$ |
| **Even** | **Odd** | 1 | $x = M;\ M > k$ |
| **Odd** | **Even** | 2 | $x = M, -M_1;\ 0 < M_1 < 1, \mathrm{M} > \mathrm{k}$ |
| **Odd** | **Odd** | 2 | $x = M, -M_1;\ 0 < M_1 < 1, \mathrm{M} > \mathrm{k}$ |

**[Table 3: Values of Roots]**

In short: The roots of (5.2.3) are

(a) One positive real number *M*, where $M > k$.

(b) One real number $-M_1$, where $0 < M_1 < 1$; when $a$ is odd.

At this point we note that total numbers of roots of (5.2.2) are $a+1$.

The roots of (5.2.2) other than above are simple complex numbers $z$ whose modulus is always less than *M*.

We express these $a$ (or $a-1$, as the case may be) complex roots in exponent form as $z_j = r_j e^{i\theta_j}$ ; where $r_j = \sqrt{a_j^{\,2} + b_j^{\,2}}$ , and $\theta_j = \tan^{-1}\left(\dfrac{b_j}{a_j}\right)$.

Also $r_j < M$ ; for all $j = 1, 2, 3, ..., a$ (or $a-1$).

In this chapter we prove that $\lim\limits_{n\to\infty} \dfrac{G_{n+a+t}}{G_{n+a}} = M^t$ ; $t$ = 1, 2, 3, … for some real number $M > k$.

## 5.3 The main result for left *k*- Fibonacci numbers:

Before we prove the main result, we first prove some intermediate results which when combined will give the main result.

Throughout we consider $t$ to be any fixed positive integer; $a$ is any integer; $M > k$ and $0 < M_1 < 1$.

**Lemma 5.3.1** When $a$ is even, $\lim_{n \to \infty} \frac{G_{n+a+t}}{G_{n+a}} = M^t$ ; $t = 1, 2, 3, \ldots$.

*Proof:* In this case the characteristic equation (5.2.2) of difference (5.2.1) has one real root $M$ such that $M > k$.

Taking into account the above discussion, by the theory of equations **[13]**, we can say that solution of the difference equation (5.2.1) is of the type

$$G_n = a_1 M^n + \sum_{j=2}^{a+1} a_j z_j^n .$$

Now, $\lim_{n \to \infty} \frac{G_{n+a+t}}{G_{n+a}} = \lim_{n \to \infty} \frac{G_n}{G_{n-t}}$

$$= \lim_{n \to \infty} \frac{a_1 M^n + \sum_{j=2}^{a+1} a_j z_j^n}{a_1 M^{n-t} + \sum_{j=2}^{a+1} a_j z_j^{n-t}}$$

$$= \lim_{n \to \infty} \frac{a_1 + \sum_{j=2}^{a+1} a_j \left(\frac{z_j}{M}\right)^n}{a_1\left(\frac{1}{M^t}\right) + \sum_{j=2}^{a+1} a_j \left(\frac{z_j}{M}\right)^{n-t} \left(\frac{1}{M^t}\right)}$$

$$= \lim_{n \to \infty} \frac{a_1 + \sum_{j=2}^{a+1} a_j \left(\frac{r_j}{M}\right)^n e^{in\theta_j}}{a_1\left(\frac{1}{M^t}\right) + \sum_{j=2}^{a+1} a_j \left(\frac{r_j}{M}\right)^{n-t} \left(\frac{1}{M^t}\right) e^{i(n-k)\theta_j}}$$

Since, $r_j < M$ , for all $j = 1, 2, 3, ...a+1$ , $\lim_{n\to\infty}\left(\frac{r_j}{M}\right)^n = 0$ and $\lim_{n\to\infty}\left(\frac{r_j}{M}\right)^{n-t} = 0$ .

Thus, $\lim_{n\to\infty}\frac{G_{n+a+t}}{G_{n+a}} = \frac{a_1}{a_1\left(\frac{1}{M^t}\right)} = M^t$ , which proves the result.

**Lemma 5.3.2** When $a$ is odd, $\lim_{n\to\infty}\frac{G_{n+a+t}}{G_{n+a}} = M^t$ ; $t = 1, 2, 3, \ldots$ .

*Proof:* When $a$ is odd, it is observed that the characteristic equation (5.2.2) has two real roots $M$ and $-M_1$ such that $M > k$ and $0 < M_1 < 1$ .

Thus we write the solution of (5.2.2) as $G_n = a_1 M^n + a_2(-M_1)^n + \sum_{j=3}^{a+1} a_j z_j^n$ .

Now, $\lim_{n\to\infty}\frac{G_{n+a+t}}{G_{n+a}} = \lim_{n\to\infty}\frac{G_n}{G_{n-t}}$

$$= \lim_{n\to\infty}\frac{a_1 M^n + a_2(-M_1)^n + \sum_{j=3}^{a+1} a_j z_j^n}{a_1 M^{n-t} + a_2(-M_1)^{n-t} + \sum_{j=3}^{a+1} a_j z_j^{n-t}}$$

$$= \lim_{n\to\infty}\frac{a_1 + a_2\left(\frac{-M_1}{M}\right)^n + \sum_{j=3}^{a+1} a_j\left(\frac{z_j}{M}\right)^n}{a_1\left(\frac{1}{M^t}\right) + a_2\left(\frac{-M_1}{M}\right)^{n-t}\left(\frac{1}{M^t}\right) + \sum_{j=3}^{a+1} a_j\left(\frac{z_j}{M}\right)^{n-t}\left(\frac{1}{M^t}\right)}$$

$$= \lim_{n\to\infty}\frac{a_1 + a_2\left(\frac{-M_1}{M}\right)^n + \sum_{j=3}^{a+1} a_j\left(\frac{r_j}{M}\right)^n e^{in\theta_j}}{a_1\left(\frac{1}{M^t}\right) + a_2\left(\frac{-M_1}{M}\right)^{n-t}\left(\frac{1}{M^t}\right) + \sum_{j=3}^{a+1} a_j\left(\frac{r_j}{M}\right)^{n-t}\left(\frac{1}{M^t}\right)e^{i(n-k)\theta_j}}$$

Since, $r_j < M$ , for all $j = 3, 4, 5, ..., a+1$ , $\lim_{n\to\infty}\left(\frac{r_j}{M}\right)^n = 0$ and $\lim_{n\to\infty}\left(\frac{r_j}{M}\right)^{n-t} = 0$ .

Also since $0 < M_1 < 1$ and $M > k$ , we have $-1 < \frac{-M_1}{M} < 0$ and $0 < \frac{M_1}{M} < 1$ .

This implies $\lim_{n \to \infty} \left(\frac{-M_1}{M}\right)^n = 0$ and $\lim_{n \to \infty} \left(\frac{M_1}{M}\right)^{n-t} = 0$.

Thus, $\lim_{n \to \infty} \frac{G_{n+a+t}}{G_{n+a}} = \frac{a_1}{a_1\left(\frac{1}{M^t}\right)} = M^t$.

**Corollary 5.3.3** For the sequence $\{G_{k,n}\}_{n=1}^{\infty}$ of Fibonacci numbers we have

$$\lim_{n \to \infty} \frac{G_{n+1}}{G_n} = \phi = \frac{1+\sqrt{5}}{2}.$$

*Proof:* By considering $a = 1$ in (5.2.1) we get the sequence of Fibonacci numbers.

Now by considering $t = 1$ in lemma 5.3.2, we have $\lim_{n\to\infty} \frac{G_{n+2}}{G_{n+1}} = M$ ,

where $M$ is the root of (5.2.2). Clearly, $x = \frac{1+\sqrt{5}}{2} = \phi$ is the root of $x^2 = x+1$.

This proves the corollary.

## 5.4 Some preliminaries of right *k*- Fibonacci numbers:

In this part of the chapter, we consider another interesting generalization of recursive formula of $F^R_{k,n}$ as $H_{n+a+1} = H_{n+a} + kH_n$; where $a$ is any positive integer. Clearly $H_n = F^R_{k,n}$ when $H_0 = 0,\ H_1 = 1$.

If we consider the *right k-Fibonacci numbers* for the case $k = 2,\ 3, \dots$ then we observe that ratio of consecutive *right k-Fibonacci numbers* converge to a fixed ratio. This fact is presented in the following table for $k = 2, 3$:

| **n** | $H_{2,n}$ | $\frac{H_{2,n+1}}{H_{2,n}}$ | $H_{3,n}$ | $\frac{H_{3,n+1}}{H_{3,n}}$ |
|---|---|---|---|---|
| **0** | 0 | | 0 | |
| **1** | 1 | 1 | 1 | 1 |
| **2** | 1 | 3 | 1 | 4 |
| **3** | 3 | 1.6666 | 4 | 1.75 |

| | | | | |
|---|---|---|---|---|
| **4** | 5 | 2.2 | 7 | 2.714285 |
| **5** | 11 | 1.90909 | 19 | 2.105263 |
| **6** | 21 | 2.047619 | 40 | 2.425 |
| **7** | 43 | 1.976744186 | 97 | 2.237113 |
| **8** | 85 | 2.011764706 | 217 | 2.341014 |
| **9** | 171 | 1.99415204 | 508 | 2.281496 |
| **10** | 341 | 2.002932551 | 1159 | 2.3149266 |
| **11** | 683 | 1.998535871 | 2683 | 2.2959374 |
| **12** | 1365 | 2.000732601 | 6160 | 2.30665585 |
| **13** | 2731 | **1.999633834** | 14209 | **2.3005841** |
| **14** | 5461 | | 32689 | |

**[Table 4: Values of Ratio]**

Here we consider the more generalized recurrence relation $H_{n+a+1} = H_{n+a} + kH_n$; where $a$ is any positive integers.

By substituting $n$ for $n+a+1$, this recurrence relation becomes

$$H_n = H_{n-1} + kH_{n-a-1} \qquad (5.4.1)$$

This is a difference equation with characteristic equation $x^n = x^{n-b} + kx^{n-a-b}$.

This is same as $$x^{a+1} = x^a + k. \qquad (5.4.2)$$

We write it as $x^a(x-1) = k$, which implies $x^a = \frac{k}{x-1}$.

Thus solving (5.2.2) is equivalent to solving the system

$$f(x) = \frac{k}{x-1}; g(x) = x^a \qquad (5.4.3)$$

This is same as finding intersection of these curves defined for $a = 1,2,3,...$

We now consider the different cases when $a$ is even or odd.

If $a$ is even, the graph of $g(x) = x^a$ is a parabolic type curve. In this case, the intersection of two curves of (5.4.3) is shown in figures 5 and 6.

Also if $a$ is odd, the graph of $g(x) = x^a$ is a curve of cubic type. In this case, the intersection of two curves of (5.4.3) is shown in figures 7 and 8.

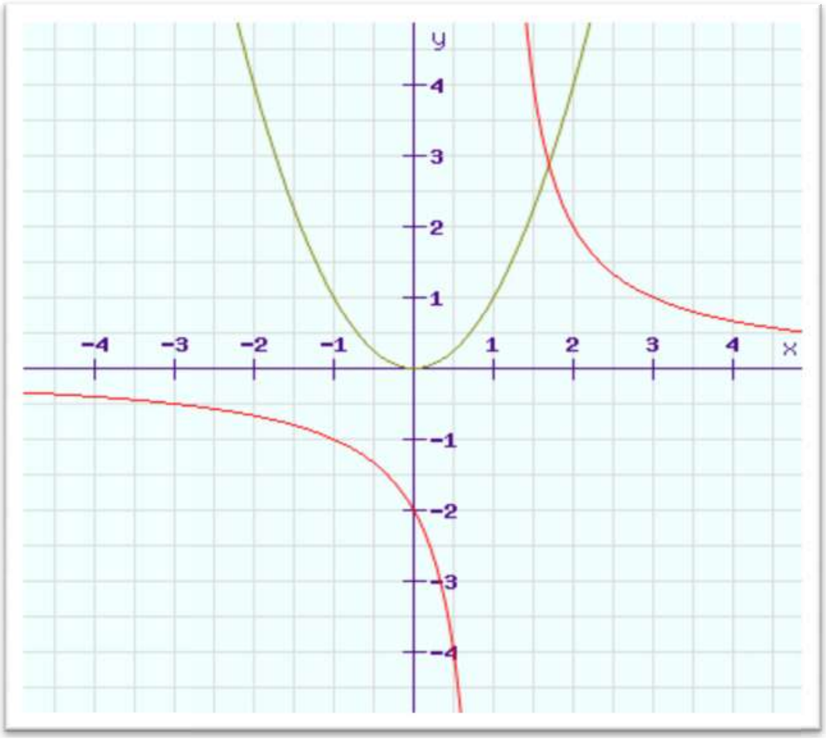



**Figure: 5 [a is even, *k* even]**

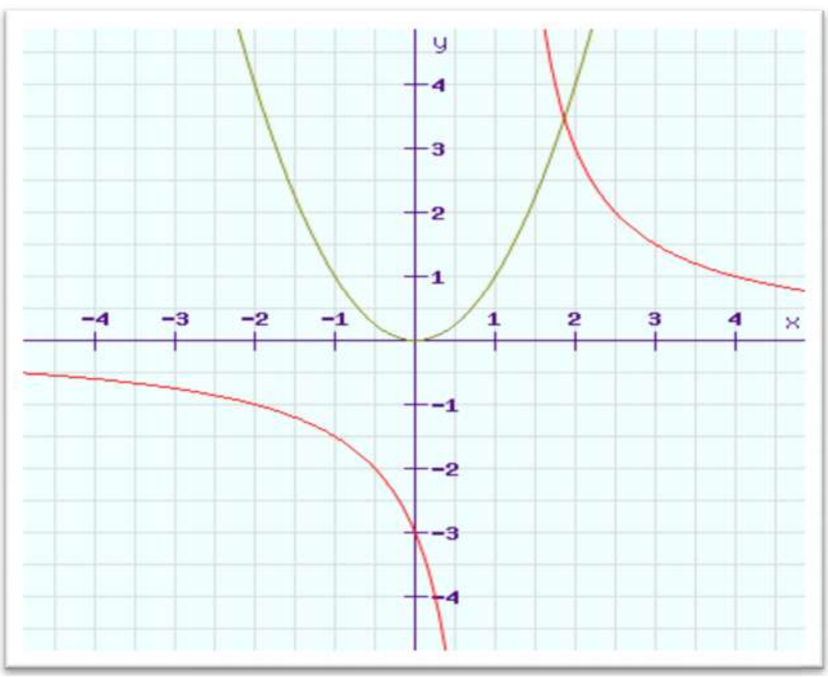



**Figure: 6 [a is even, *k* odd]**

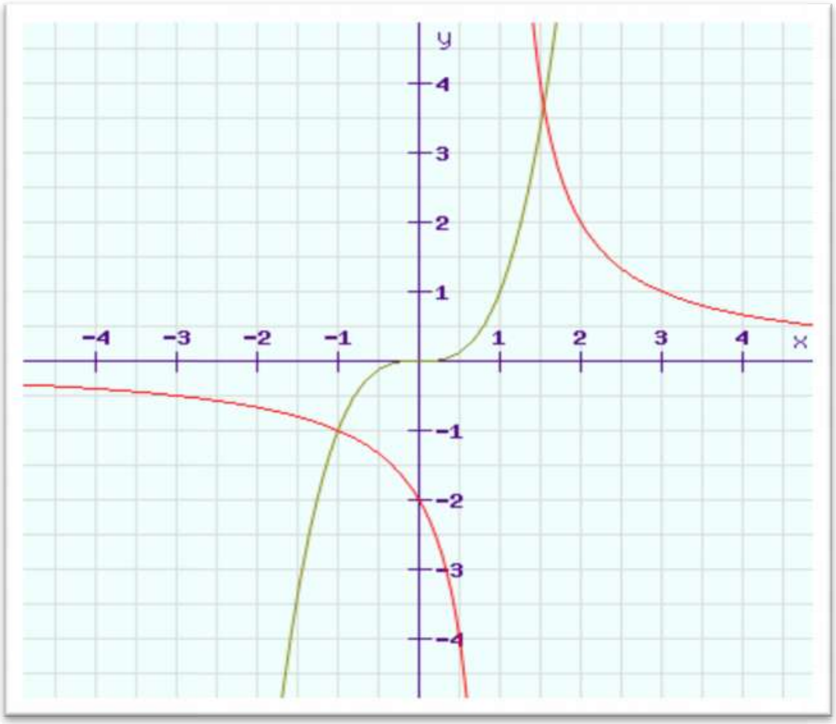



**Figure: 7 [a is odd, *k* even]**

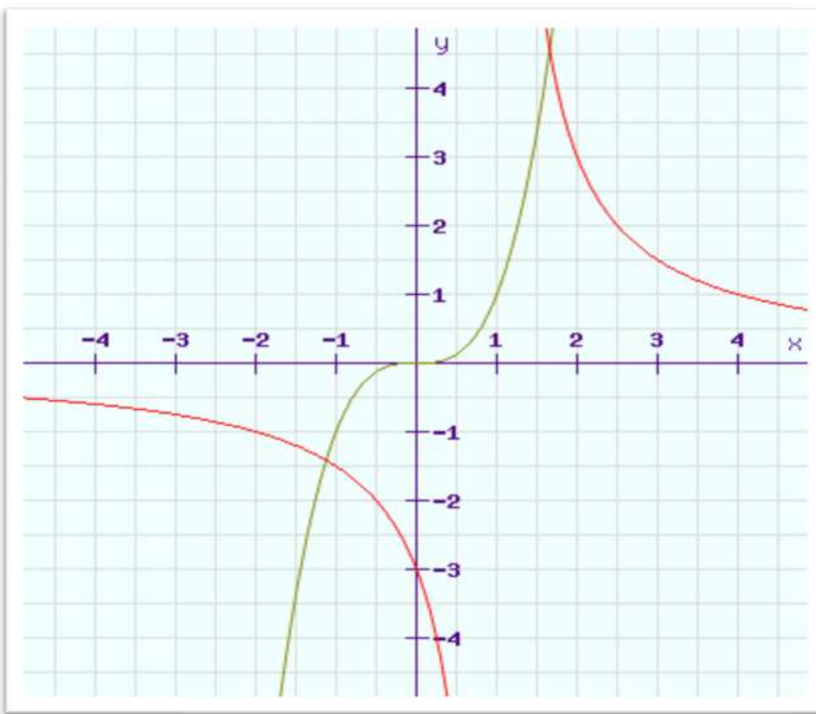



**Figure: 8 [*a* is odd, *k* odd]**

It is clear from these figures that when $a$ is even both $f(x)$ and $g(x)$ are symmetric about the y-axis. In this case, system (5.4.3) has one real root $x = M$ ; for some real number M such that, $M > 1$.When $a$ is odd, system (5.4.3) has two real roots $x = M, -M_1$, where $M > M_1 \geq 1$.

We summarize this in the following table.

| **a** | ***k*** | **Number of intersecting Points** | **Roots** |
|---|---|---|---|
| **Even** | **Even** | 1 | $x = M; M > 1$ |
| **Even** | **Odd** | 1 | $x = M; M > 1$ |
| **Odd** | **Even** | 2 | $x = M, -M_1;\ M > M_1 \geq 1$ |
| **Odd** | **Odd** | 2 | $x = M, -M_1;\ M > M_1 \geq 1$ |

**[Table 5: Values of Roots]**

In short: The roots of (5.4.3) are

(a) One positive real number *M*, where $M > 1$.

(b) One real number $-M_1$, where $M > M_1 \geq 1$; when $a$ is odd.

At this point we note that total numbers of roots of (5.4.3) are $a+1$.

The roots of (5.4.2) other than above are simple complex numbers $z$ whose modulus is always less than *M*.

We express these $a$ (or $a-1$, as the case may be) complex roots in exponent form as $z_j = r_j e^{i\theta_j}$; where $r_j = \sqrt{a_j^{\,2} + b_j^{\,2}}$, and $\theta_j = \tan^{-1}\left(\frac{b_j}{a_j}\right)$ and also $r_j < M$; for all $j = 1, 2, 3, ..., a$ (or $a-1$).

In fact we prove that $\lim_{n \to \infty} \frac{G_{n+a+t}}{G_{n+a}} = M^t$ ; $t$ = 1, 2, 3, …; for some real number $M > 1$.

## 5.5. The main result for right *k*-Fibonacci numbers:

Before we prove the main result, we first prove some intermediate results which when combined will give the main result.

Throughout we consider $t$ to be any fixed positive integer; $a$ is any integer; $M > 1$ and $M > M_1 \geq 1$.

**Lemma 5.5.1** When $a$ is even, $\lim_{n \to \infty} \frac{H_{n+a+t}}{H_{n+a}} = M^t$ ; $t = 1, 2, 3, \ldots$.

*Proof:* Since $a$ is even, the characteristic equation (5.4.2) of difference (5.4.1) has one real root $M$ such that $M > 1$.

Taking into account the above discussion, by the theory of equations **[13]** we can say that solution of the difference equation (5.4.1) is as under:

$$H_n = a_1 M^n + \sum_{j=2}^{a+1} a_j z_j^n .$$

Now, $\lim_{n \to \infty} \frac{H_{n+a+t}}{H_{n+a}} = \lim_{n \to \infty} \frac{H_n}{H_{n-t}}$

$$= \lim_{n \to \infty} \frac{a_1 M^n + \sum_{j=2}^{a+1} a_j z_j^n}{a_1 M^{n-t} + \sum_{j=2}^{a+1} a_j z_j^{n-t}}$$

$$= \lim_{n \to \infty} \frac{a_1 + \sum_{j=2}^{a+1} a_j \left(\frac{z_j}{M}\right)^n}{a_1 \left(\frac{1}{M^t}\right) + \sum_{j=2}^{a+1} a_j \left(\frac{z_j}{M}\right)^{n-t} \left(\frac{1}{M^t}\right)}$$

$$= \lim_{n \to \infty} \frac{a_1 + \sum_{j=2}^{a+1} a_j \left(\frac{r_j}{M}\right)^n e^{in\theta_j}}{a_1 \left(\frac{1}{M^t}\right) + \sum_{j=2}^{a+1} a_j \left(\frac{r_j}{M}\right)^{n-t} \left(\frac{1}{M^t}\right) e^{i(n-k)\theta_j}}$$

Since, $r_j < M$ , for all $j = 1,2,3,\ldots a+1$ , $\lim_{n \to \infty} \left(\frac{r_j}{M}\right)^n = 0$ and $\lim_{n \to \infty} \left(\frac{r_j}{M}\right)^{n-t} = 0$ .

Thus, $\lim_{n \to \infty} \frac{H_{n+a+t}}{H_{n+a}} = \frac{a_1}{a_1 \left(\frac{1}{M^t}\right)} = M^t$ , which proves the result.

**Lemma 5.5.2** When $a$ is odd , $\lim_{n\to\infty}\frac{H_{n+a+t}}{H_{n+a}}=M^t$ ; $t = 1, 2, 3, \ldots$.

*Proof:* When $a$ is odd. It is seen that the characteristic equation (5.4.2) has two real roots $M$ and $-M_1$ such that $M > M_1 \geq 1$.

Thus we write the solution of (5.2.2) as $H_n = a_1M^n + a_2(-M_1)^n + \sum_{j=3}^{a+1} a_j z_j^n$ .

Now, $\lim_{n\to\infty}\frac{H_{n+a+t}}{H_{n+a}} = \lim_{n\to\infty}\frac{H_n}{H_{n-t}}$

$$= \lim_{n\to\infty} \frac{a_1M^n + a_2(-M_1)^n + \sum_{j=3}^{a+1} a_j z_j^n}{a_1M^{n-t} + a_2(-M_1)^{n-t} + \sum_{j=3}^{a+1} a_j z_j^{n-t}}$$

$$= \lim_{n\to\infty} \frac{a_1 + a_2\left(\frac{-M_1}{M}\right)^n + \sum_{j=3}^{a+1} a_j\left(\frac{z_j}{M}\right)^n}{a_1\left(\frac{1}{M^t}\right) + a_2\left(\frac{-M_1}{M}\right)^{n-t}\left(\frac{1}{M^t}\right) + \sum_{j=3}^{a+1} a_j\left(\frac{z_j}{M}\right)^{n-t}\left(\frac{1}{M^t}\right)}$$

$$= \lim_{n\to\infty} \frac{a_1 + a_2\left(\frac{-M_1}{M}\right)^n + \sum_{j=3}^{a+1} a_j\left(\frac{r_j}{M}\right)^n e^{in\theta_j}}{a_1\left(\frac{1}{M^t}\right) + a_2\left(\frac{-M_1}{M}\right)^{n-t}\left(\frac{1}{M^t}\right) + \sum_{j=3}^{a+1} a_j\left(\frac{r_j}{M}\right)^{n-t}\left(\frac{1}{M^t}\right)e^{i(n-k)\theta_j}}.$$

Since, $r_j < M$ , for all $j = 3,4,5,\ldots,a+1$, $\lim_{n\to\infty}\left(\frac{r_j}{M}\right)^n = 0$ and $\lim_{n\to\infty}\left(\frac{r_j}{M}\right)^{n-t} = 0$.

Also since $M > M_1 \geq 1$, we have $-1 < \frac{-M_1}{M} \leq 0$ and $0 < \frac{M_1}{M} < 1$.

This implies $\lim_{n\to\infty}\left(\frac{-M_1}{M}\right)^n = 0$ and $\lim_{n\to\infty}\left(\frac{M_1}{M}\right)^{n-t} = 0$.

Thus, $\lim_{n\to\infty}\frac{H_{n+a+t}}{H_{n+a}} = \frac{a_1}{a_1\left(\frac{1}{M^t}\right)} = M^t$ .

It is now mere a formality to state the main result of this chapter.

**Corollary5.5.3** For the sequence $\{H_{k,n}\}_{n=1}^{\infty}$ of Fibonacci numbers we have

$$\lim_{n\to\infty} \frac{H_{n+1}}{H_n} = \phi = \frac{1+\sqrt{5}}{2}.$$

*Proof:* By considering $a = 1$ in (5.4.1) we get the sequence of Fibonacci numbers.

Now by considering $t = 1$ in lemma 5.5.2, we have $\lim_{n\to\infty} \frac{H_{n+2}}{H_{n+1}} = M$ , where $M$ is the root of (5.4.2).

Clearly, $x = \frac{1+\sqrt{5}}{2} = \phi$ is the root of $x^2 = x+1$.

This proves the corollary.

## Scope:

If we consider the more generalized recurrence relations as $G_{n+b} = kG_{n+a} + G_n$ and $H_{n+b} = H_{n+a} + kH_n$; where $a,b$ are positive integers such that $b > a$ , then there are three real parameters $a,b$ and $k$ which are to be considered. In this case also we can find the corresponding 'golden proportion' for the whole class of generalized Fibonacci sequence. There is a great scope of work possible in this case.

## Conclusions:

New generalized k- Fibonacci and associated k- Fibonacci sequences has been introduced and deducted their identities and results.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# *References*

[1] Alvaro H. Salas: *About k-Fibonacci Numbers and their Associated Numbers*, International Mathematical Forum, Vol. 6 (2011), Issue 50, page 2473-2479.

[2] Apostol T. M.: *Introduction to Analytic Number Theory*, Narosa Pub. House, N. Delhi, 1998.

[3] Arvadia Mansukh P., Shah Devbhadra V.: *Right k-Fibonacci sequence and related identities*, International Research Journal of Mathematics, Engineering & IT, Volume-2, Issue-4 (2015), 25-39.

[4] Arvadia Mansukh P., Shah Devbhadra V.: *Left k-Fibonacci Sequence and Related Identities*, Journal Club for Applied Sciences, V–2, I–1(2015), 20-26.

[5] Atkins J., Geist R.: "*Fibonacci numbers and computer algorithms*", College Math. J.18 (1987), 328-336.

[6] Burton D. M.: *Elementary Number Theory*, Tata Mc Graw–Hill Pub. Co. Ltd., N. Delhi, $6^{th}$Edn, 2007.

[7] Carlitz L.: *Fibonacci Representations*, the Fibonacci Quarterly, 6(4), 1968, 193–220.

[8] Cennet Bolat, Hasan Kose: *On the properties of k-Fibonacci Numbers*, Int. J. Contemp. Math. Sciences, Vol. 5, (2010), No. 22, 1097–1105.

[9] De Villiers M.: *A Fibonacci generalization and its dual*, International Journal of Mathematics Education, Science and Technology, 31, (2000), 464–473.

[10] Dickson L. E.: *History of the Theory of Numbers*-Vol. II: Diophantine Analysis, Chelsea Pub. Co., New York, 1971.

[11] Diwan Daksha M., Shah Devbhadra V.: *Explicit and recursive formula for class of generalized Fibonacci sequence*, Proceedings of 19th Annual Cum $4^{th}$ International Conference of GAMS, (2014), 104 – 108.

[12] Dunlap R. A.: *The golden ratio and Fibonacci numbers*, Revised Ed., World Scientific, Singapore, 1997, pp. 61.

[13] Elaydi S. N.: *An introduction to difference equations*, Springer–Verlag, New York, 2nd Edn., 1999.

[14] Falcon S: *A simple proof of an interesting Fibonacci generalization*, International Journal of Mathematics Education, Science and Technology, 35, No. 2, (2004), 259–261.

[15] Falcon S., Plaza, A: *Binomial Transforms of the k-Fibonacci Sequence*, International Journal of Non-linear Sciences and Numerical Simulation, Vol. 10, Issue (11) (2009) 1527-1538.

[16] Falcon S., Plaza, A :*On the Fibonacci k-numbers*, Chaos, Solitons & Fractals, 32 (5) (2007) 1615–24.

[17] Falcon S., Plaza, A :*The k-Fibonacci sequence and Pascal 2-triangle,* Chaos, Solitons & Fractals (33) (1) (2007) 38–49.

[18] Fibonacci numbers and the golden section:

http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/

[19] Geller S. P.: *A computer investigation of a property of the Fibonacci sequence*, The Fibonacci Quarterly 1:2, April (1963), pp. 84.

[20] Gupta V. K., Yashawant K. Panwar, Shikhwal O.: *Generalized Fibonacci Sequences*, Theoretical Mathematics & Applications, Vol. 2, No. 2, 2012, 115-124.

[21] Horadam A. F.: *Basic properties of certain generalized sequence of numbers*, The Fibonacci Quarterly, 3,1965, 161–176.

[22] Johnson M. B., Fielder D. C.: *The number of representations of N using distinct Fibonacci numbers counted by recursive formulas*, The Fibonacci Quarterly, 37(1) (1999), 47-60.

[23] Klein S. T.: *Combinatorial representation of generalized Fibonacci numbers*, Fibonacci Quart.29 (1991) 124-131.

[24] Koshy Thomas: *Fibonacci and Lucas Numbers with applications*, John Wiley and Sons, Inc., N. York, 2001.

[25] Lee G. Y., Lee S. G., H. G. Shin: *On the k-generalized Fibonacci matrix $Q_k$*, Linear Algebra Appl. 251(1997), 73-88.

[26] Lee G. Y., Lee S. G., Kim J. S., Shin H. K.: *The Binet formula and representations of k-generalized Fibonacci numbers*, Fibonacci Quart. 39(2001), no. 2, 158-164.

[27] Marcia Edson, Omer Yayenie: *A new generalization of Fibonacci sequence and extended Binet's formula*, Integers, Volume 9, Issue 6, (2009), 639–654.

[28] Miles E. P. Jr.: *Generalized Fibonacci numbers and associated matrices*, Amer. Math. Monthly, 67(1960), 745-757.

[29] Panwar Yashwant K., Rathore G. P. S., Chawla Richa: *On the k- Fibonacci- Like Numbers,* Turkish Journal of Analysis and Number Theory, 2014, Vol. 2, No. 1, 9-12.

[30] Panwar Yashwant K., Singh Bijendra, Gupta V. K.: *Identities of common Factors of Generalized Fibonacci, Jacobsthal and Jacobsthal-Lucas Numbers,* Applied Mathematics and Phsics, 2013, Vol. 1, No. 4, 126-128.

[31] Panwar Yashwant K., Singh Mamta: *k-Generalized Fibonacci Numbers,* Applied Mathematics and Phsics, 2014, Vol. 2, No. 1, 10-12.

[32] Pond J. C.: *Generalized Fibonacci Summations*, Fibonacci Quart. 6 (1968), 97-108.

[33] Renault Marc: *Properties of the Fibonacci sequence under various moduli*, Master's Thesis, Wake Forest University, 1996.

[34] Robinson D. W.: *The Fibonacci matrix modulo m*, The Fibonacci Quarterly, 1 (1963), 29-36.

[35] Shah A. P.: *Fibonacci sankhyao (in Gujarati)*, Gujarat Granth Nirman Board, Ahmedabad (India), 1971.

[36] Shah D. V., *Non-existence of GCD property for the generalized tribonacci numbers*, International Journal of Physical, Chemical and Mathematical Sciences, Vol. 1, No. 1, 2012.

[37] Shah D. V., Mehta D. A.: *Golden proportions for the generalized Tribonacci numbers*, Int. Jr. Math. Educ. Sci. Technol., 40(6) (2009), 837-842.

[38] Singh B., Sikhwal O., Bhatnagar S.: *Fibonacci-like sequence and its properties*, Int. J. Contemp-Math. Sciences, Vol.5, No.18, 2010, 857–868.

[39] Stakhov A.: *The golden section and modern harmony mathematics; Application of Fibonacci numbers*, Kluwer Academic Pub., Vol. 7, (1998), 393-399.

[40] Stakhov A, Rozin B.: *Theory of Binet formulas for Fibonacci and Lucas p-numbers*. Chaos, Solitons & Fractals2006;27(5):1162–77.

[41] Stakhov A.: *Fundamentals of a new kind of mathematics based on the golden section*, Chaos, Solitons & Fractals2006;27(5):1124–46.

[42] Suck J. B., Schreiber M., Häussler P.: "*Quasicrystals: An Introduction to Structure, Physical Properties, and Applications*", Springer-Verlag, Berlin Heidelberg, 2002.

[43] Vajda Steven: *Fibonacci and Lucas numbers and the golden section*, Ellis Horwood Ltd., Chichester, England, 1989.

[44] Vandana Patel, Shah D. V.: *Generalized Fibonacci sequence and its properties*, International Journal of Physics and Mathematical Science, Vol. 4(2), April-June 2014, 118-124.

[45] Walton J. E., Horadam A. F.: *Some further identities for the generalized Fibonacci sequence* $\{H_n\}$, Fibonacci Quart. 12(1974), 272-280.

[46] Zeitlin D.: *On summation formulas and identities for Fibonacci numbers*, The Fibonacci Quarterly, 5(1) (1967), 1-43.

# Appendix-I

Computer Program to obtain terms of left k- Fibonacci sequence $\{F_n\}$ is presented here using the programming language MATLAB (R2008a).

```
clear all
clc
syms k;
f0 = 0; f1 = 1;
n = input('Enter the number of terms:');
disp(['F0',' = ',num2str(f0)]);
disp(['F1',' = ',num2str(f1)]);
for i=2:n
    F = k*f1 + f0;
Fn = expand(F);
disp(['F',num2str(i),' = ',char(Fn)]);
    f0=f1; f1 = Fn;
end
```

# Appendix-II

Computer Program to obtain terms of right k- Fibonacci sequence $\{G_n\}$ is presented here using the programming language MATLAB (R2008a).

```
clear all
clc
syms k;
G0 = 0; G1 = 1;
n = input('Enter the number of terms:');
disp(['G0',' = ',num2str(G0)]);
disp(['G1',' = ',num2str(G1)]);
for i=2:n
    G = G1 + k*G0;
Gn = expand(G);
disp(['G',num2str(i),' = ',char(Gn)]);
    G0=G1; G1 = Gn;
end
```

# Appendix-III

Computer Program to obtain terms of associated left k- Fibonacci sequence $\{A_n\}$ is presented here using the programming language MATLAB (R2008a).

```
clear all

clc

syms k;

A0 = 1; A1 = 1;

n = input('Enter the number of terms:');

disp(['A0',' = ',num2str(A0)]);

disp(['A1',' = ',num2str(A1)]);

for i=2:n

    A = k*A1 + A0;

    An = expand(A);

disp(['A',num2str(i),' = ',char(An)]);

    A0=A1; A1 = An;

end
```

# Appendix-IV

Computer Program to obtain terms of associated right k- Fibonacci sequence $\{B_n\}$ is presented here using the programming language MATLAB (R2008a).

```
clear all

clc

syms k;

B0 = 1/k; B1 = 1;

n = input('Enter the number of terms:');

disp(['B0',' = ',char(B0)]);

disp(['B1',' = ',num2str(B1)]);

for i=2:n

    B = B1 + k*B0;

Bn = expand(B);

disp(['B',num2str(i),' = ',char(Bn)]);

    B0=B1; B1 = Bn;

End
```

## *Publication Details of Mansukh P. Arvadia*

**Faculty of Science** **Subject: Mathematics**

| S. No. | Name of Authors | Title | Name of Journal / Book | ISSN / ISBN | Volume & Year | Number |
|---|---|---|---|---|---|---|
| 1 | Arvadia Mansukh P. and Dr. Shah Devbhadra V. | Right *k*- Fibonacci Sequence And Related Identities | International Research Journal of Mathematics, Engineering & IT | 2349-0322 | 2, April 2015 Issue - 4 | 25-39 |
| 2 | Arvadia Mansukh P. and Dr. Shah Devbhadra V. | Left *k*- Fibonacci Sequence and Related Identities | Journal Club for Applied Sciences | 2394 – 1731 | V – 2, I – 1, 2015 | 20-26 |
| 3 | Arvadia Mansukh P | Left k- Fibonacci Numbers and its Binet's Formula | International Journal of Engineering Science and Computing | 2250-1371 | V –62, I – 9, Sep 2016 | 2144-2148 |